WEAK CONVERGENCE OF PARTIAL MAXIMA PROCESSES IN THE M_1 TOPOLOGY

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ABSTRACT. It is known that for a sequence of independent and identically distributed random variables (X_n) the regular variation condition is equivalent to weak convergence of partial maxima $M_n = \max\{X_1, \ldots, X_n\}$, appropriately scaled. A functional version of this is known to be true as well, the limit process being an extremal process, and the convergence takes place in the space of càdlàg functions endowed with the Skorohod J_1 topology. We first show that weak convergence of partial maxima M_n holds also for a class of weakly dependent sequences under the joint regular variation condition. Then using this result we obtain a corresponding functional version for the processes of partial maxima $M_n(t) = \bigvee_{i=1}^{\lfloor nt \rfloor} X_i$, $t \in [0, 1]$, but with respect to the Skorohod M_1 topology, which is weaker than the more usual J_1 topology. We also show that the M_1 convergence generally can not be replaced by the J_1 convergence. Applications of our main results to moving maxima, squared GARCH and ARMAX processes are also given.

1. INTRODUCTION

Let (X_i) be a strictly stationary sequence of nonnegative random variables and denote by $M_n = \max\{X_i : i = 1, \ldots, n\}$, $n \geq 1$, its accompanying sequence of partial maxima. It is well known that in the i.i.d. case weak convergence of M_n is equivalent to the regular variation property of X_1. More precisely, let (a_n) be a sequence of positive real numbers such that

$$n \, \mathrm{P}(X_1 > a_n) \to 1 \qquad \text{as } n \to \infty, \tag{1.1}$$

and μ a measure of the form

$$\mu(dx) = \alpha x^{-\alpha-1} 1_{(0,\infty)}(x) \, dx$$

for some $\alpha > 0$. Then

$$n \, \mathrm{P}\left(\frac{X_1}{a_n} \in \cdot \right) \xrightarrow{v} \mu(\,\cdot\,) \tag{1.2}$$

is equivalent to

$$\frac{M_n}{a_n} \xrightarrow{d} Y_0,$$

where Y_0 is a random variable with Fréchet distribution

$$\mathrm{P}(Y_0 \leq x) = e^{-\mu((x,\infty))} = e^{-x^{-\alpha}}, \qquad x \geq 0$$

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(see for example Resnick [24], Proposition 7.1). The arrow " $\xrightarrow{v}$ " in (1.2) denotes vague convergence of measures on $\mathbb{E} = (0, \infty]$. The regular variation property (1.2) is equivalent to

$$\mathrm{P}(X_1 > x) = x^{-\alpha} L(x), \qquad x > 0,$$

where $L(\,\cdot\,)$ is a slowly varying function at ∞.

In the i.i.d. case the regular variation property (1.2) is also equivalent to the functional convergence of stochastic processes of partial maxima of (X_n), i.e.

$$M_n(\,\cdot\,) = \bigvee_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} \xrightarrow{d} Y_0(\,\cdot\,) \tag{1.3}$$

in $D[0,1]$, the space of real-valued càdlàg functions on $[0,1]$, with the Skorohod J_1 topology, where $Y_0(\,\cdot\,)$ is an extremal process with exponent measure μ, therefore with marginal distributions

$$\mathrm{P}(Y_0(t) \le x) = e^{-t\mu((x,\infty))} = e^{-tx^{-\alpha}}, \quad x \ge 0, \, t \in [0,1].$$

This result was proved by Lamperti [16] (see also Resnick [24], Proposition 7.2). For convenience we can put $M_n(t) = X_1/a_n$ (or $M_n(t) = 0$) for $t \in [0, 1/n)$. Weissman [27] studied generalizations of this result to random variables which need not be identically distributed. As for the dependent case, Adler [1] obtained J_1 extremal functional convergence with the weak dependence condition similar to "asymptotic independence" condition introduced by Leadbetter [17] (see also Leadbetter [18]). J_1 functional convergence of sample extremal processes of moving averages was obtained by Davis and Resnick [9] with the noise having regularly varying tail probabilities, and by Jordanova [14] with the noise in the Weibull max-domain of attraction. In the recent years different functional limit theorems for extremal processes subordinated to random time were obtained, see for instance Silvestrov and Teugels [25]; Meerschaert and Stoev [19].

In this paper, under the properties of weak dependence and joint regular variation with index $\alpha \in (0, \infty)$ for the sequence (X_n), we investigate the asymptotic distributional behavior of extremes M_n and processes $M_n(\,\cdot\,)$. Since we study extremes of random processes, nonnegativity of random variables X_n in reality is not a restrictive assumption. First, we introduce the essential ingredients about regular variation and weak dependence in Section 2. In Section 3 we prove the so called timeless result on weak convergence of scaled extremes M_n, based on a point process convergence obtained by Davis and Hsing [7]. Using this result and a limit theorem derived by Basrak et al. [5] for a certain time-space point processes, in Section 4 we prove a functional limit theorem for processes of partial maxima $M_n(\,\cdot\,)$ in the space $D[0,1]$ endowed with the Skorohod M_1 topology, which is weaker than the frequently used J_1 topology. The used methods are partly based on the work of Basrak et al. [5] for partial sums. Finally, in Section 5 we discuss several examples of stationary sequences covered by our functional limit theorem, and show that the M_1 convergence, in general, can not be replaced by the J_1 convergence.

2. Preliminaries on regular variation, point processes and weak dependence

In this section we introduce the basic notions and results on regular variation and point processes that will be required for the results in the following sections.

Multivariate regular variation or regular variation on $\mathbb{R}_+^d = [0,\infty)^d$ for random vectors is typically formulated in terms of vague convergence on $\mathbb{E}^d = [0,\infty]^d \setminus \{\mathbf{0}\}$. The topology on $\mathbb{E}^d$ is chosen so that a set $B \subseteq \mathbb{E}^d$ has compact closure if and only if it is bounded away from zero, that is, if there exists $u > 0$ such that $B \subseteq \mathbb{E}_u^d = \{\mathbf{x} \in \mathbb{E}^d : \|\mathbf{x}\| > u\}$. Here $\|\cdot\|$ denotes the max-norm on $\mathbb{R}_+^d$, i.e. $\|\mathbf{x}\| = \max\{x_i : i = 1,\dots,d\}$ where $\mathbf{x} = (x_1,\dots,x_d) \in \mathbb{R}_+^d$.

The vector $\boldsymbol{\xi}$ with values in $\mathbb{R}_+^d$ is (multivariate) regularly varying with index $\alpha > 0$ if there exists a random vector $\boldsymbol{\Theta}$ on the unit sphere $\mathbb{S}_+^{d-1} = \{\mathbf{x} \in \mathbb{R}_+^d : \|\mathbf{x}\| = 1\}$ in $\mathbb{R}_+^d$, such that for every $u \in (0,\infty)$

$$\frac{\mathrm{P}(\|\boldsymbol{\xi}\| > ux,\, \boldsymbol{\xi}/\|\boldsymbol{\xi}\| \in \cdot\,)}{\mathrm{P}(\|\boldsymbol{\xi}\| > x)} \xrightarrow{w} u^{-\alpha}\,\mathrm{P}(\boldsymbol{\Theta} \in \cdot\,) \tag{2.1}$$

as $x \to \infty$, where the arrow "$\xrightarrow{w}$" denotes weak convergence of finite measures. Regular variation can be expressed in terms of vague convergence of measures on $\mathcal{B}(\mathbb{E}^d)$:

$$n\,\mathrm{P}(a_n^{-1}\boldsymbol{\xi} \in \cdot\,) \xrightarrow{v} \mu(\,\cdot\,),$$

where (a_n) is a sequence of positive real numbers tending to infinity and μ is a non-null Radon measure on $\mathcal{B}(\mathbb{E}^d)$.

We say that a strictly stationary $\mathbb{R}_+$–valued process (ξ_n) is *jointly regularly varying* with index $\alpha \in (0,\infty)$ if for any nonnegative integer k the k-dimensional random vector $\boldsymbol{\xi} = (\xi_1,\dots,\xi_k)$ is multivariate regularly varying with index α.

Theorem 2.1 in Basrak and Segers [6] provides a convenient characterization of joint regular variation: it is necessary and sufficient that there exists a process $(Y_n)_{n\in\mathbb{Z}}$ with $\mathrm{P}(Y_0 > y) = y^{-\alpha}$ for $y \geq 1$ such that as $x \to \infty$,

$$\big((x^{-1}\xi_n)_{n\in\mathbb{Z}} \,\big|\, \xi_0 > x\big) \xrightarrow{\text{fidi}} (Y_n)_{n\in\mathbb{Z}}, \tag{2.2}$$

where "$\xrightarrow{\text{fidi}}$" denotes convergence of finite-dimensional distributions. The process (Y_n) is called the *tail process* of (ξ_n).

Let (X_i) be a strictly stationary sequence of nonnegative random variables and assume it is jointly regularly varying with index $\alpha \in (0,\infty)$. The property of joint regular variation is a corner stone in obtaining the weak convergence of point processes N_n given by

$$N_n = \sum_{i=1}^{n} \delta_{X_i/a_n}, \qquad n \in \mathbb{N},$$

with a_n as in (1.1). These point processes play a fundamental role in obtaining the limit theorem for scaled extremes M_n. The following time-space point processes

$$N_n^* = \sum_{i=1}^{n} \delta_{(i/n,\, X_i/a_n)}, \qquad n \in \mathbb{N}, \tag{2.3}$$

will be used in obtaining the functional limit theorem for processes of partial maxima $M_n(\,\cdot\,)$.

To control the dependence in the sequence (X_n) we first have to assume that clusters of large values of X_n do not last for too long.

Condition 2.1. *There exists a sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$ and such that for every $u > 0$,*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathrm{P}\bigg(\max_{m \le |i| \le r_n} X_i > ua_n \,\bigg|\, X_0 > ua_n \bigg) = 0. \tag{2.4}$$

Under the finite-cluster Condition 2.1 the following value

$$\theta = \lim_{r\to\infty} \lim_{x\to\infty} \mathrm{P}\big(M_r \le x \,\big|\, X_0 > x\big) \tag{2.5}$$

is strictly positive, and it is equal to the extremal index of the sequence (X_n) (see Basrak and Segers [6]). The extremal index can be interpreted as the reciprocal mean cluster size of large exceedances (cf. Hsing et al. [13]). Clustering of extreme values occurs when $\theta < 1$.

The weak dependence condition appropriate for our considerations is the mixing condition called $\mathcal{A}'(a_n)$ which is slightly stronger than the condition $\mathcal{A}(a_n)$ introduced in Davis and Hsing [7]. Condition $\mathcal{A}'(a_n)$ is implied by the strong mixing property, see Krizmani'c [15]. Recall that a sequence of random variables (ξ_n) is strongly mixing if $\alpha(n) \to 0$ as $n \to \infty$, where

$$\alpha(n) = \sup\{|\,\mathrm{P}(A \cap B) - \mathrm{P}(A)\,\mathrm{P}(B)| : A \in \mathcal{F}_1^j, B \in \mathcal{F}_{j+n}^{\infty}, j = 1, 2, \ldots\}$$

and $\mathcal{F}_k^l = \sigma(\{\xi_i : k \le i \le l\})$ for $1 \le k \le l \le \infty$.

Condition 2.2 ($\mathcal{A}'(a_n)$)**.** *There exists a sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$ and such that for every nonnegative continuous function f on $[0,1] \times \mathbb{E}$ with compact support, denoting $k_n = \lfloor n/r_n \rfloor$, as $n \to \infty$,*

$$\mathrm{E}\bigg[\exp\bigg\{-\sum_{i=1}^{n} f\bigg(\frac{i}{n}, \frac{X_i}{a_n}\bigg)\bigg\}\bigg] - \prod_{k=1}^{k_n} \mathrm{E}\bigg[\exp\bigg\{-\sum_{i=1}^{r_n} f\bigg(\frac{kr_n}{n}, \frac{X_i}{a_n}\bigg)\bigg\}\bigg] \to 0. \tag{2.6}$$

Under joint regular variation and Conditions 2.1 and 2.2, by Theorem 2.7 in Davis and Hsing [7] we obtain that the point processes N_n, $n \in \mathbb{N}$, converge in distribution to some N, which by Theorem 2.3 and Corollary 2.4 in Davis and Hsing [7] has the following cluster representation

$$N \stackrel{d}{=} \sum_i \sum_j \delta_{P_i Q_{ij}}, \tag{2.7}$$

where $\sum_{i=1}^{\infty} \delta_{P_i}$ is a Poisson process with intensity measure ν given by $\nu(dy) = \theta\alpha y^{-\alpha-1} 1_{(0,\infty)}(y)\,dy$, and $\sum_{j=1}^{\infty} \delta_{Q_{ij}}$, $i \ge 1$, are i.i.d. point processes on $[0,1]$ whose points satisfy $\sup_j Q_{ij} = 1$, and all point processes are mutually independent. The distribution of the point process $\sum_{j=1}^{\infty} \delta_{Q_{ij}}$ is described in Davis and Hsing [7].

Conditions 2.1 and 2.2, by Theorem 2.3 in Basrak et al. [5], also imply convergence in distribution of the point process N_n^* on the set $[0,1] \times \mathbb{E}_u$ for every $u \in (0,\infty)$, where $\mathbb{E}_u = (u, \infty]$. More precisely, under the above conditions, for every $u \in (0,\infty)$, as $n \to \infty$,

$$N_n^*\Big|_{[0,1]\times\mathbb{E}_u} \xrightarrow{d} N^{(u)} = \sum_i \sum_j \delta_{(T_i^{(u)}, uZ_{ij})}\Big|_{[0,1]\times\mathbb{E}_u} \tag{2.8}$$

in $[0,1] \times \mathbb{E}_u$, where $\sum_i \delta_{T_i^{(u)}}$ is a homogeneous Poisson process on $[0,1]$ with intensity $\theta u^{-\alpha}$, and $(\sum_j \delta_{Z_{ij}})_i$ is an i.i.d. sequence of point processes in $\mathbb{E}$, independent of $\sum_i \delta_{T_i^{(u)}}$, and with common distribution equal to the distribution of

$$\Big(\sum_{n \in \mathbb{Z}} \delta_{Y_n} \,\Big|\, \sup_{i \leq -1} Y_i \leq 1 \Big),$$

where (Y_n) is the tail process of the sequence (X_n).

For a detailed discussion on joint regular variation and dependence conditions 2.1 and 2.2 we refer to Basrak et al. [5], Section 3.4.

3. Weak convergence of partial maxima M_n

In this section we establish convergence of the partial maxima M_n for a class of weakly dependent sequences. Precisely, let (X_n) be a strictly stationary sequence of nonnegative random variables, jointly regularly varying with index $\alpha \in (0, \infty)$ and assume Conditions 2.1 and 2.2 hold. Then by (2.7) it holds that, as $n \to \infty$,

$$N_n = \sum_{i=1}^{n} \delta_{X_i / a_n} \xrightarrow{d} N = \sum_i \sum_j \delta_{P_i Q_{ij}},$$

where (a_n) is chosen as in (1.1). Denote by $\mathbf{M}_p(\mathbb{E})$ the space of Radon point measures on $\mathbb{E}$ equipped with the vague topology. Recall $M_n = \bigvee_{i=1}^{n} X_i$.

Theorem 3.1. *Let (X_n) be a strictly stationary sequence of nonnegative random variables, jointly regularly varying with index $\alpha \in (0, \infty)$. Suppose that Conditions 2.1 and 2.2 hold. Then, as $n \to \infty$,*

$$\frac{M_n}{a_n} \xrightarrow{d} M,$$

where the limit M is a Fréchet random variable with

$$\mathrm{P}(M \leq x) = e^{-\theta x^{-\alpha}}, \qquad x > 0.$$

Proof. Define $M = \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} P_i Q_{ij}$, and let $\epsilon > 0$ be arbitrary. The mapping $T_\epsilon \colon \mathbf{M}_p(\mathbb{E}) \to \mathbb{R}$ defined by

$$T_\epsilon \Big(\sum_{i=1}^{\infty} \delta_{x_i} \Big) = \bigvee_{i=1}^{\infty} x_i 1_{\{x_i \in [\epsilon, \infty)\}}$$

is continuous on the set $\Lambda_\epsilon = \{\eta \in \mathbf{M}_p(\mathbb{E}) : \eta(\{\epsilon\}) = 0\}$ (cf. Resnick [23], page 214). Since N has no fixed atoms (see Lemma 2.1 in Davis and Hsing [7]), i.e. $\mathrm{P}(N \in \Lambda_\epsilon) = 1$, using the continuous mapping theorem we obtain

$$M_n[\epsilon, \infty) = T_\epsilon(N_n) \xrightarrow{d} T_\epsilon(N) = M[\epsilon, \infty) \qquad \text{as } n \to \infty, \tag{3.1}$$

with the notation

$$M_n B = a_n^{-1} \bigvee_{i=1}^{n} X_i 1_{\{a_n^{-1} X_i \in B\}},$$

and

$$M B = \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{\infty} P_i Q_{ij} 1_{\{P_i Q_{ij} \in B\}}$$

for any Borel set B in $\mathbb{R}$. Obviously

$$M[\epsilon, \infty) \to M(0, \infty) = M \tag{3.2}$$

almost surely as $\epsilon \to 0$. If we show that

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \mathrm{P}(|M_n[\epsilon, \infty) - M_n(0, \infty)| > \delta) = 0 \tag{3.3}$$

for any $\delta > 0$, then by Theorem 3.5 in Resnick [24] we will have $M_n(0,\infty) \xrightarrow{d} M(0,\infty)$, i.e. $a_n^{-1} M_n \xrightarrow{d} M$ as $n \to \infty$.

Since for arbitrary real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ the following inequality

$$\Big| \bigvee_{i=1}^{n} x_i - \bigvee_{i=1}^{n} y_i \Big| \le \bigvee_{i=1}^{n} |x_i - y_i| \tag{3.4}$$

holds, note that

$$|M_n[\epsilon, \infty) - M_n(0, \infty)| \le M_n(0, \epsilon).$$

Take an arbitrary $s > \alpha$. Then using stationarity and Markov's inequality we get the bound

$$\begin{aligned} \mathrm{P}(M_n(0,\epsilon) > \delta) \quad &\le \quad n\,\mathrm{P}\left(\frac{X_1}{a_n} 1_{\{X_1 < \epsilon a_n\}} > \delta\right) \le \frac{n}{\delta^s a_n^s}\,\mathrm{E}(X_1^s 1_{\{X_1 < \epsilon a_n\}}) \\ &= \quad \frac{\epsilon^s}{\delta^s} \cdot n\,\mathrm{P}(X_1 > a_n) \cdot \frac{\mathrm{P}(X_1 > \epsilon a_n)}{\mathrm{P}(X_1 > a_n)} \cdot \frac{\mathrm{E}(X_1^s 1_{\{X_1 < \epsilon a_n\}})}{\epsilon^s a_n^s\,\mathrm{P}(X_1 > \epsilon a_n)}. \end{aligned} \tag{3.5}$$

Since the distribution of X_1 is regularly varying with index α, it follows immediately that

$$\frac{\mathrm{P}(X_1 > \epsilon a_n)}{\mathrm{P}(X_1 > a_n)} \to \epsilon^{-\alpha}$$

as $n \to \infty$. By Karamata's theorem

$$\lim_{n \to \infty} \frac{\mathrm{E}(X_1^s\, 1_{\{X_1 < \epsilon a_n\}})}{\epsilon^s a_n^s\,\mathrm{P}(X_1 > \epsilon a_n)} = \frac{\alpha}{s - \alpha}.$$

Thus from (3.5), taking into account relation (1.1), we get

$$\limsup_{n \to \infty} \mathrm{P}(M_n(0,\epsilon) > \delta) \le \delta^{-s} \frac{\alpha}{s - \alpha} \epsilon^{s - \alpha}.$$

Letting $\epsilon \to 0$, since $s - \alpha > 0$, we finally obtain

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \mathrm{P}(M_n(0,\epsilon) > \delta) = 0,$$

and relation (3.3) holds. Therefore $a_n^{-1} M_n \xrightarrow{d} M$ as $n \to \infty$.

From the representation in (2.7) and the fact that $\sup_j Q_{ij} = 1$ we obtain the distribution of the limit M,

$$\mathrm{P}(M \le x) = \mathrm{P}\left(\bigvee_{i=1}^{\infty} P_i \le x\right) = \mathrm{P}\left(\sum_i \delta_{P_i}(x, \infty) = 0\right) = e^{-\nu(x,\infty)} = e^{-\theta x^{-\alpha}}$$

for $x > 0$. □

4. Functional convergence of partial maxima processes $M_n(\,\cdot\,)$

In this section we show the convergence of the partial maxima processes $M_n(\,\cdot\,)$ to an extremal process in the space $D[0,1]$ equipped with the Skorohod M_1 topology. Similar to the case of partial sum processes in Basrak et al. [5] we first represent the partial maxima process $M_n(\,\cdot\,)$ as the image of the time-space point process N_n^* under a certain maximum functional. Then, using certain continuity properties of this functional, the continuous mapping theorem and the standard "finite dimensional convergence plus tightness" procedure we transfer the weak convergence of N_n^* in (2.8) to weak convergence of $M_n(\,\cdot\,)$.

Recall the definition of the M_1 topology. For $x \in D[0,1]$ the *completed graph* of x is the set

$$\Gamma_x = \{(t,z) \in [0,1] \times \mathbb{R} : z = \lambda x(t-) + (1-\lambda)x(t) \text{ for some } \lambda \in [0,1]\},$$

where $x(t-)$ is the left limit of x at t. Besides the points of the graph $\{(t,x(t)) : t \in [0,1]\}$, the completed graph of x also contains the vertical line segments joining $(t,x(t))$ and $(t,x(t-))$ for all discontinuity points t of x. We define an *order* on the graph Γ_x by saying that $(t_1,z_1) \le (t_2,z_2)$ if either (i) $t_1 < t_2$ or (ii) $t_1 = t_2$ and $|x(t_1-) - z_1| \le |x(t_2-) - z_2|$. A *parametric representation* of the completed graph Γ_x is a continuous nondecreasing function (r,u) mapping $[0,1]$ onto Γ_x, with r being the time component and u being the spatial component. Let $\Pi(x)$ denote the set of parametric representations of the graph Γ_x. For $x_1, x_2 \in D[0,1]$ define

$$d_{M_1}(x_1,x_2) = \inf\{\|r_1 - r_2\|_{[0,1]} \vee \|u_1 - u_2\|_{[0,1]} : (r_i,u_i) \in \Pi(x_i), i = 1,2\},$$

where $\|x\|_{[0,1]} = \sup\{|x(t)| : t \in [0,1]\}$ and $a \vee b = \max\{a,b\}$. d_{M_1} is a metric on $D[0,1]$, and the induced topology is called the Skorohod M_1 topology. This topology is weaker than the more frequently used Skorohod J_1 topology. For more discussion of the M_1 topology we refer to Whitt [28], sections 12.3-12.5.

Fix $0 < v < u < \infty$. Define the maximum functional

$$\phi^{(u)} \colon \mathbf{M}_p([0,1] \times \mathbb{E}_v) \to D[0,1]$$

by

$$\phi^{(u)}\Big(\sum_i \delta_{(t_i,\,x_i)}\Big)(t) = \bigvee_{t_i \le t} x_i \, 1_{\{u < x_i < \infty\}}, \qquad t \in [0,1],$$

where the supremum of an empty set may be taken, for convenience, to be 0. Note that $\phi^{(u)}$ is well defined because $[0,1] \times \mathbb{E}_u$ is a relatively compact subset of $[0,1] \times \mathbb{E}_v$. Indeed, for every $\eta \in \mathbf{M}_p([0,1] \times \mathbb{E}_v)$ it holds that $\eta([0,1] \times \mathbb{E}_u) < \infty$, and this immediately yields $\phi^{(u)}(\eta) \in D[0,1]$. The space $\mathbf{M}_p([0,1] \times \mathbb{E}_v)$ of Radon point measures on $[0,1] \times \mathbb{E}_v$ is equipped with the vague topology and $D[0,1]$ is equipped with the M_1 topology. Let

$$\Lambda = \{\eta \in \mathbf{M}_p([0,1] \times \mathbb{E}_v) : \eta(\{0,1\} \times \mathbb{E}_u) = \eta([0,1] \times \{u,\infty\}) = 0\}.$$

Observe that elements of Λ are Radon point measures that have no atoms on the border of $[0,1] \times \mathbb{E}_u$. Then the point process $N^{(v)}$ defined in (2.8) almost surely belongs to the set Λ, see Lemma 3.1 in Basrak et al. [5]. Now we will show that $\phi^{(u)}$ is continuous on the set Λ.

Lemma 4.1. *The maximum functional $\phi^{(u)} \colon \mathbf{M}_p([0,1] \times \mathbb{E}_v) \to D[0,1]$ is continuous on the set Λ, when $D[0,1]$ is endowed with the Skorohod M_1 topology.*

Proof. Take an arbitrary $\eta \in \Lambda$ and suppose that $\eta_n \xrightarrow{v} \eta$ in $\mathbf{M}_p([0,1] \times \mathbb{E}_v)$. We will show that $\phi^{(u)}(\eta_n) \to \phi^{(u)}(\eta)$ in $D[0,1]$ according to the M_1 topology. Since the set $[0,1] \times \mathbb{E}_u$ is relatively compact in $[0,1] \times \mathbb{E}_v$, there exists a nonnegative integer $k = k(\eta)$ such that

$$\eta([0,1] \times \mathbb{E}_u) = k < \infty.$$

By assumption, η does not have any atoms on the border of the set $[0,1] \times \mathbb{E}_u$. Hence, by Lemma 7.1 in Resnick [24], there exists a positive integer n_0 such that for all $n \geq n_0$ it holds that

$$\eta_n([0,1] \times \mathbb{E}_u) = k.$$

If $k = 0$ there is nothing to prove, so assume $k \geq 1$, and let (t_i, x_i) for $i = 1, \ldots, k$ be the atoms of η in $[0,1] \times \mathbb{E}_u$. By the same lemma, the k atoms $(t_i^{(n)}, x_i^{(n)})$ of η_n in $[0,1] \times \mathbb{E}_u$ (for $n \geq n_0$) can be labelled in such a way that for every $i \in \{1, \ldots, k\}$ we have

$$(t_i^{(n)}, x_i^{(n)}) \to (t_i, x_i) \qquad \text{as } n \to \infty.$$

In particular, for any $\delta > 0$ we can find a positive integer $n_\delta \geq n_0$ such that for all $n \geq n_\delta$,

$$|t_i^{(n)} - t_i| < \delta \quad \text{and} \quad |x_i^{(n)} - x_i| < \delta \qquad \text{for } i = 1, \ldots, k. \tag{4.1}$$

Let the sequence

$$0 < \tau_1 < \tau_2 < \ldots < \tau_p < 1$$

be such that the sets $\{\tau_1, \ldots, \tau_p\}$ and $\{t_1, \ldots, t_k\}$ coincide. Since η can have several atoms with the same time coordinate, it always holds that $p \leq k$. Put $\tau_0 = 0$, $\tau_{p+1} = 1$, and take

$$0 < r < \frac{1}{2} \min_{0 \leq i \leq p} |\tau_{i+1} - \tau_i|.$$

For any $t \in [0,1] \setminus \{\tau_1, \ldots, \tau_p\}$ we can find $\delta \in (0, u)$ such that

$$\delta < r \quad \text{and} \quad \delta < \min_{1 \leq i \leq p} |t - \tau_i|.$$

Then relation (4.1), for $n \geq n_\delta$, implies that $t_i^{(n)} \leq t$ is equivalent to $t_i \leq t$, and we obtain

$$|\phi^{(u)}(\eta_n)(t) - \phi^{(u)}(\eta)(t)| = \bigg| \bigvee_{t_i^{(n)} \leq t} x_i^{(n)} - \bigvee_{t_i \leq t} x_i \bigg| \leq \bigvee_{t_i \leq t} |x_i^{(n)} - x_i| < \delta.$$

Therefore

$$\lim_{n \to \infty} |\phi^{(u)}(\eta_n)(t) - \phi^{(u)}(\eta)(t)| < \delta,$$

and if we let $\delta \to 0$, it follows that $\phi^{(u)}(\eta_n)(t) \to \phi^{(u)}(\eta)(t)$ as $n \to \infty$. Note that the functions $\phi^{(u)}(\eta)$ and $\phi^{(u)}(\eta_n)$ $(n \geq n_\delta)$ are monotone. Since, by Corollary 12.5.1 in Whitt [28], M_1 convergence for monotone functions is equivalent to pointwise convergence in a dense subset of points plus convergence at the endpoints, we obtain that $d_{M_1}(\phi^{(u)}(\eta_n), \phi^{(u)}(\eta)) \to 0$ as $n \to \infty$, i.e. $\phi^{(u)}$ is continuous at η. $\square$

In the sequel we will show that the functional $\phi^{(u)}$ is not continuous on the set Λ when $D[0,1]$ is endowed with the Skorohod J_1 topology (see Remark 4.2).

The theorem below gives conditions under which the partial maxima processes of a strictly stationary, jointly regularly varying sequence of nonnegative random variables satisfies a functional limit theorem with an extremal process as a limit. Extremal processes can be defined by Poisson processes in the following way. Let

$\xi = \sum_k \delta_{(t_k, j_k)}$ be a Poisson process on $(0,\infty) \times \mathbb{E}$ with mean measure $\lambda \times \nu$, where λ is the Lebesgue measure. The extremal process $\widetilde{M}(\,\cdot\,)$ generated by ξ is defined by

$$\widetilde{M}(t) = \sup\{j_k : t_k \le t\}, \qquad t > 0.$$

The distribution function of $\widetilde{M}(t)$ is of the form

$$\mathrm{P}(\widetilde{M}(t) \le x) = e^{-t\nu(x,\infty)}$$

for $t > 0$ (cf. Resnick [22]). The measure ν is called the exponent measure.

The convergence in the theorem takes place in the space $D[0,1]$ endowed with the Skorohod M_1 topology. In the proof of the theorem we will need a characterization of M_1 convergence for random processes which is due to Skorohod. Put

$$M(x_1, x_2, x_3) = \begin{cases} 0, & \text{if } x_2 \in [x_1, x_3], \\ \min\{|x_2 - x_1|, |x_3 - x_2|\}, & \text{otherwise,} \end{cases}$$

(note that $M(x_1,x_2,x_3)$ is the distance from x_2 to $[x_1,x_3]$) and introduce the M_1 oscillation $\omega_\delta(x)$ of a function $x \in D[0,1]$ by

$$\omega_\delta(x) = \sup_{\substack{t_1 \le t \le t_2 \\ 0 \le t_2 - t_1 \le \delta}} M(x(t_1), x(t), x(t_2)),$$

for $\delta > 0$. Then the following corollary of Theorems 3.2.1 and 3.2.2 in Skorohod [26] holds.

Proposition 4.2. *Let $Z_n(\,\cdot\,)$ be processes in $D[0,1]$ whose finite dimensional distributions converge to those of a process $Z(\,\cdot\,)$ which is a.s. continuous at $t = 0$ and $t = 1$. Then $Z_n(\,\cdot\,)$ converges in distribution to $Z(\,\cdot\,)$ in $D[0,1]$ with respect to the Skorohod M_1 topology if and only if for every $\epsilon > 0$,*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathrm{P}(\omega_\delta(Z_n(\,\cdot\,)) > \epsilon) = 0. \tag{4.2}$$

Remark 4.1. The statement of Proposition 4.2 remains valid if the M_1 topology is replaced by the J_1 topology, and the M_1 oscillation $\omega_\delta(\,\cdot\,)$ is replaced by the J_1 oscillation $\omega'_\delta(\,\cdot\,)$ defined by

$$\omega'_\delta(x) = \sup_{\substack{t_1 \le t \le t_2 \\ 0 \le t_2 - t_1 \le \delta}} \min\{|x(t) - x(t_1)|, |x(t_2) - x(t)|\},$$

for $x \in D[0,1]$ and $\delta > 0$ (see Skorohod [26]).

Theorem 4.3. *Let (X_n) be a strictly stationary sequence of nonnegative random variables, jointly regularly varying with index $\alpha \in (0,\infty)$. Suppose that Conditions 2.1 and 2.2 hold. Then the partial maxima stochastic process*

$$M_n(t) = \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{X_i}{a_n}, \qquad t \in [0,1],$$

satisfies

$$M_n(\,\cdot\,) \xrightarrow{d} \widetilde{M}(\,\cdot\,), \qquad n \to \infty,$$

in $D[0,1]$ endowed with the M_1 topology, where $\widetilde{M}(\,\cdot\,)$ is an extremal process with exponent measure $\nu(x,\infty) = \theta x^{-\alpha}$, $x > 0$, with θ given by (2.5).

Proof. Using the techniques from the proof of Theorem 3.4 in Basrak et al. [5] we obtain that the point process

$$\widehat{N}^{(u)} = \sum_i \delta_{(T_i^{(u)}, u \bigvee_j Z_{ij} 1_{\{Z_{ij}>1\}})},$$

is a Poisson process with mean measure $\lambda \times \nu^{(u)}$, where the measure $\nu^{(u)}$ is defined by

$$\nu^{(u)}(x,\infty) = u^{-\alpha} \ \mathrm{P}\bigg(u \bigvee_{i \geq 0} Y_i \, 1_{\{Y_i > 1\}} > x, \ \sup_{i \leq -1} Y_i \leq 1 \bigg), \qquad x > 0,$$

with (Y_i) being the tail process of the sequence (X_i).

Consider now $0 < u < v$ and

$$\phi^{(u)}(N_n^* \,|\, _{[0,1] \times \mathbb{E}_u})(\,\cdot\,) = \phi^{(u)}(N_n^* \,|\, _{[0,1] \times \mathbb{E}_v})(\,\cdot\,) = \bigvee_{i/n \leq \, \cdot} \frac{X_i}{a_n} 1_{\big\{ \frac{X_i}{a_n} > u \big\}},$$

which by Lemma 4.1 and the continuous mapping theorem converges in distribution in $D[0,1]$ under the M_1 metric to

$$\phi^{(u)}(N^{(v)})(\,\cdot\,) = \phi^{(u)}(N^{(v)} \,|\, _{[0,1] \times \mathbb{E}_u})(\,\cdot\,).$$

Since by the definition of the process $N^{(u)}$ in (2.8) it holds that $N^{(u)} \stackrel{d}{=} N^{(v)}|_{[0,1]\times\mathbb{E}_u}$, the last expression above is equal in distribution to

$$\phi^{(u)}(N^{(u)})(\,\cdot\,) = \bigvee_{T_i^{(u)} \leq \, \cdot} \ \bigvee_j u Z_{ij} 1_{\{Z_{ij}>1\}}.$$

But since $\phi^{(u)}(N^{(u)}) = \phi^{(u)}(\widehat{N}^{(u)}) \stackrel{d}{=} \phi^{(u)}(\widetilde{N}^{(u)})$, where

$$\widetilde{N}^{(u)} = \sum_i \delta_{(T_i, \, K_i^{(u)})}$$

is a Poisson process (or Poisson random measure) with mean measure $\lambda \times \nu^{(u)}$, we obtain

$$M_n^{(u)}(\,\cdot\,) := \bigvee_{i=1}^{\lfloor n \, \cdot \rfloor} \frac{X_i}{a_n} 1_{\big\{ \frac{X_i}{a_n} > u \big\}} \xrightarrow{d} M^{(u)}(\,\cdot\,) := \bigvee_{T_i \leq \, \cdot} K_i^{(u)} \quad \text{as } n \to \infty, \tag{4.3}$$

in $D[0,1]$ under the M_1 metric. Note that the limiting process $M^{(u)}(\,\cdot\,)$ is an extremal process with exponent measure $\nu^{(u)}$, since

$$\mathrm{P}(M^{(u)}(t) \leq x) = \mathrm{P}(\widetilde{N}^{(u)}((0,t] \times (x,\infty)) = 0) = e^{-t\nu^{(u)}(x,\infty)}$$

for $t \in [0,1]$ and $x > 0$.

Since the function $\pi \colon D[0,1] \to \mathbb{R}$ defined by $\pi(x) = x(1)$ is continuous (see Theorem 12.5.1 (iv) in Whitt [28]), from (4.3) using the continuous mapping theorem, we obtain

$$M_n^{(u)}(1) \xrightarrow{d} M^{(u)}(1) \qquad \text{as } n \to \infty. \tag{4.4}$$

If we now apply the notation from the proof of Theorem 3.1, we see that $M_n^{(u)}(1) = M_n(u,\infty)$. Therefore comparing (3.1) and (4.4) we conclude that $M^{(u)}(1) \stackrel{d}{=} M(u,\infty)$. Further, from (3.2) it follows that $M^{(u)}(1) \xrightarrow{d} M$ as $u \to 0$. Therefore taking into account the distribution of M we conclude that $e^{-\nu^{(u)}(x,\infty)} \to$

$e^{-\nu(x,\infty)}$ for all $x > 0$ that are continuity points of the distribution of M, where $\nu(dy) = \theta\alpha y^{-\alpha-1} 1_{(0,\infty)}(y)\, dy$. Hence

$$\nu^{(u)}(x,\infty) \to \nu(x,\infty) \qquad \text{as } u \to 0, \tag{4.5}$$

for every continuity point x of $\nu(\,\cdot\,,\infty)$.

Now we show that the finite dimensional distributions of $M^{(u)}(\,\cdot\,)$ converge, as u tends to zero, to the finite dimensional distributions of an extremal process $\widetilde{M}(\,\cdot\,)$ generated by a Poisson process $\sum_i \delta_{(T_i,K_i)}$ with mean measure $\lambda \times \nu$, i.e. $\widetilde{M}(t) = \bigvee_{T_i \leq t} K_i$, $t \in [0,1]$. Since $M^{(u)}(\,\cdot\,)$ is an extremal process, its finite dimensional distributions are of the form

$$\begin{aligned} &\mathrm{P}(M^{(u)}(t_1) \leq x_1, \ldots, M^{(u)}(t_k) \leq x_k) \\ &\qquad = \quad e^{-t_1 \nu^{(u)}(\bigwedge_{i=1}^{k} x_i,\infty)} \cdot e^{-(t_2-t_1)\nu^{(u)}(\bigwedge_{i=2}^{k} x_i,\infty)} \cdot \ldots \cdot e^{-(t_k-t_{k-1})\nu^{(u)}(x_k,\infty)}, \end{aligned}$$

for $0 \leq t_1 < t_2 < \ldots < t_k \leq 1$ and positive real numbers $x_1, \ldots, x_k$ (see Resnick [22], Section 2.3). Letting $u \to 0$ and using (4.5) we immediately obtain that the right hand side in the last equation above converges (in the continuity points $x_1, \ldots, x_k$ of $\nu(\,\cdot\,,\infty)$) to

$$e^{-t_1(\bigwedge_{i=1}^{k} x_i,\infty)} \cdot e^{-(t_2-t_1)\nu(\bigwedge_{i=2}^{k} x_i,\infty)} \cdot \ldots \cdot e^{-(t_k-t_{k-1})\nu(x_k,\infty)}.$$

But since this limit is in fact $\mathrm{P}(\widetilde{M}(t_1) \leq x_1, \ldots, \widetilde{M}(t_k) \leq x_k)$, we conclude that the finite dimensional distributions of $M^{(u)}(\,\cdot\,)$ converge to the finite dimensional distributions of $\widetilde{M}(\,\cdot\,)$ as $u \to 0$.

Since $\widetilde{M}(\,\cdot\,)$ is constructed from a Poisson process, using its properties one can easily obtain that $\widetilde{M}(\,\cdot\,)$ is a.s. continuous at $t = 0$ and $t = 1$. In order to obtain M_1 convergence of $M^{(u)}(\,\cdot\,)$ to $\widetilde{M}(\,\cdot\,)$ as $u \to 0$, according to Proposition 4.2, we need only to show (4.2), i.e

$$\lim_{\delta \to 0} \limsup_{u \to 0} \mathrm{P}(\omega_\delta(M^{(u)}(\,\cdot\,)) > \epsilon) = 0.$$

Note that since $M^{(u)}(\,\cdot\,)$ is increasing, for $t_1 \leq t \leq t_2$ it holds that $M^{(u)}(t_1) \leq M^{(u)}(t) \leq M^{(u)}(t_2)$, which implies $M(M^{(u)}(t_1), M^{(u)}(t), M^{(u)}(t_2)) = 0$. Hence $\omega_\delta(M^{(u)}) = 0$, and (4.2) holds. Therefore $M^{(u)}(\,\cdot\,) \xrightarrow{d} \widetilde{M}(\,\cdot\,)$ in $D[0,1]$ with the M_1 topology.

So far we obtained $M_n^{(u)}(\,\cdot\,) \xrightarrow{d} M^{(u)}(\,\cdot\,)$ as $n \to \infty$, and $M^{(u)}(\,\cdot\,) \xrightarrow{d} \widetilde{M}(\,\cdot\,)$ as $u \to 0$. If we show

$$\lim_{u \to 0} \limsup_{n \to \infty} \mathrm{P}(d_{M_1}(M_n(\,\cdot\,), M_n^{(u)}(\,\cdot\,)) > \epsilon) = 0,$$

for every $\epsilon > 0$, then by Theorem 3.5 in Resnick [24] we will have, as $n \to \infty$,

$$M_n(\,\cdot\,) \xrightarrow{d} \widetilde{M}(\,\cdot\,)$$

in $D[0,1]$ with the M_1 topology. Take an arbitrary (and fixed) $\epsilon > 0$. Using the fact that the Skorohod M_1 metric on $D[0,1]$ is bounded above by the uniform metric on $D[0,1]$ and relation (3.4) we obtain

$$\begin{aligned}
&\mathrm{P}(d_{M_1}(M_n(\,\cdot\,), M_n^{(u)}(\,\cdot\,)) > \epsilon) \\
&\qquad \leq \mathrm{P}\bigg(\sup_{t \in [0,1]} |M_n(t) - M_n^{(u)}(t)| > \epsilon \bigg) \\
&\qquad = \mathrm{P}\bigg(\sup_{t \in [0,1]} \bigg| \bigvee_{j=1}^{\lfloor nt \rfloor} \frac{X_j}{a_n} - \bigvee_{j=1}^{\lfloor nt \rfloor} \frac{X_j}{a_n} 1_{\big\{ \frac{X_j}{a_n} > u \big\}} \bigg| > \epsilon \bigg) \\
&\qquad \leq \mathrm{P}\bigg(\sup_{t \in [0,1]} \bigg| \bigvee_{j=1}^{\lfloor nt \rfloor} \frac{X_j}{a_n} 1_{\big\{ \frac{X_j}{a_n} \leq u \big\}} \bigg| > \epsilon \bigg) \\
&\qquad \leq \mathrm{P}\bigg(\bigvee_{j=1}^{n} \frac{X_j}{a_n} 1_{\big\{ \frac{X_j}{a_n} \leq u \big\}} > \epsilon \bigg)
\end{aligned}$$

Note that the last term above is equal to zero for $u \in (0, \epsilon)$. Hence

$$\lim_{u \to 0} \limsup_{n \to \infty} \mathrm{P}(d_{M_1}(M_n(\,\cdot\,), M_n^{(u)}(\,\cdot\,)) > \epsilon) = 0,$$

and this concludes the proof. □

Remark 4.2. The M_1 convergence in Theorem 4.3 in general can not be replaced by the J_1 convergence. This is shown in Example 5.1.

The problem in our proof if we consider the J_1 topology is Lemma 4.1, which in this case does not hold. To see this, fix $u > 0$ and define

$$\eta_n = \delta_{(\frac{1}{2} - \frac{1}{n}, 2u)} + \delta_{(\frac{1}{2}, 3u)} \qquad \text{for } n \geq 3.$$

Then $\eta_n \xrightarrow{v} \eta$, where

$$\eta = \delta_{(\frac{1}{2}, 2u)} + \delta_{(\frac{1}{2}, 3u)}.$$

For $t_n = \frac{1}{2} - \frac{1}{n}$ and every strictly increasing continuous function $\lambda \colon [0,1] \to [0,1]$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$, we have

$$\phi^{(u)}(\eta_n)(t_n) = 2u \qquad \text{and} \qquad \phi^{(u)}(\eta)(\lambda(t_n)) \in \{0, 3u\}.$$

Therefore for every $n \geq 3$,

$$\|\phi^{(u)}(\eta_n) - \phi^{(u)}(\eta \circ \lambda)\|_{[0,1]} \geq |\phi^{(u)}(\eta_n)(t_n) - \phi^{(u)}(\eta)(\lambda(t_n))| \geq u,$$

and by the definition of the J_1 metric d_{J_1} (see for example Resnick [23], Section 4.4.1) we obtain

$$d_{J_1}(\phi^{(u)}(\eta_n), \phi^{(u)}(\eta \circ \lambda)) \geq u,$$

which means that $\phi^{(u)}(\eta_n)$ does not converge to $\phi^{(u)}(\eta)$ in the J_1 topology, i.e. the maximum functional $\phi^{(u)}$ is not continuous at η with respect to the Skorohod J_1 topology. Since $\eta \in \Lambda$ we conclude that $\phi^{(u)}$ is not continuous on the set Λ.

In our case the J_1 topology is inappropriate as the partial maxima process may exhibit rapid successions of jumps within temporal clusters of large values, collapsing in the limit to a single jump. In other words the J_1 convergence could hold only if extreme values do not cluster. Since our conditions do not prohibit clustering of extremes, the J_1 convergence fails to hold, and hence the weaker M_1 topology has to be used.

Remark 4.3. Theorems 3.1 and 4.3 can be extended to real-valued random variables, in the sense that convergence in distribution of $a_n^{-1} M_n$ and $M_n(\,\cdot\,)$ can be derived analogously with the use of absolute values of the variables X_i, Z_{ij} and Y_i in appropriate places. But with the methods used for positive random variables, one can not obtain a explicit form for the distribution of the limits, i.e. the distribution function of M and the exponent measure of $\widetilde{M}(\,\cdot\,)$.

5. Examples

We give three examples of time series that satisfy all conditions in Theorems 3.1 and 4.3, namely joint regular variation property and Conditions 2.1 and 2.2. Hence for these processes we obtain convergence of partial maxima M_n and functional convergence of partial maxima processes $M_n(\,\cdot\,)$. We also identify the distribution of the corresponding limits M and $\widetilde{M}(\,\cdot\,)$ by indicating explicitly the extremal index θ. Recall $\mathrm{P}(M \le x) = e^{-\theta x^{-\alpha}}$ and $\mathrm{P}(\widetilde{M}(t) \le x) = e^{-t\theta x^{-\alpha}}$ for $x > 0$ and $t > 0$.

Example 5.1. (Moving maxima) Consider the finite order moving maxima defined by

$$X_n = \max_{i=0,\ldots,m} \{c_i Z_{n-i}\}, \qquad n \in \mathbb{Z},$$

where $m \in \mathbb{N}$, $c_0, \ldots, c_m$ are nonnegative constants such that at least c_0 and c_m are not equal to 0 and Z_i, $i \in \mathbb{Z}$, are i.i.d. unit Fréchet random variables, i.e. $\mathrm{P}(Z_i \le x) = e^{-1/x}$ for $x > 0$. Hence Z_i is regularly varying with index $\alpha = 1$. Take a sequence of positive real numbers (a_n) such that

$$n\,\mathrm{P}(Z_1 > a_n) \to 1 \qquad \text{as } n \to \infty.$$

Then every X_i is also regularly varying with index $\alpha = 1$. Assume also (without loss of generality) that $\sum_{i=0}^{m} c_i = 1$. Then $n\,\mathrm{P}(X_1 > a_n) \to 1$ as $n \to \infty$. Since the sequence (X_n) is m–dependent, it is also strongly mixing, and therefore the mixing Condition 2.2 holds. By the same property, for $s > m$ we have

$$\begin{aligned} \mathrm{P}\bigg(\max_{s \le |i| \le r_n} X_i > u a_n \,\bigg|\, X_0 > u a_n \bigg) &= \mathrm{P}\bigg(\max_{s \le |i| \le r_n} X_i > u a_n \bigg) \\ &\le \frac{2 r_n}{n} \cdot n\,\mathrm{P}(X_1 > u a_n). \end{aligned}$$

Note that the expression on the right hand side in the above inequality converges to 0 as $n \to \infty$, and hence Condition 2.1 also holds. By an application of Theorem 2.3 in Meinguet [20] we obtain that the sequence (X_n) is jointly regularly varying with index $\alpha = 1$. The extremal index of the sequence (X_n) is given by $\theta = \max_{0 \le i \le m}\{c_i\}$ (see Ancona-Navarrete and Tawn [2] and Meinguet [20]).

In the rest of the example we show that the M_1 convergence in Theorem 4.3 in general can not be replaced by the J_1 convergence. We use, with appropriate modifications, the procedure of Avram and Taqqu [3] in the proof of their Theorem 1. For simplicity take $m = 2$. Then we have $X_n = \max\{c_0 Z_n, c_1 Z_{n-1}\}$ and

$$M_n(t) = \bigvee_{i=1}^{\lfloor nt \rfloor} \frac{X_i}{a_n}, \qquad t \in [0,1].$$

By Remark 4.1 it suffices to prove

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathrm{P}(\omega_\delta'(M_n(\,\cdot\,)) > \epsilon) > 0 \tag{5.1}$$

for some $\epsilon > 0$.

Assume $c_1 > c_0$. Let $i' = i'(n)$ be the index at which $\max_{1 \le i \le n-1} Z_i$ is obtained. Fix $\epsilon > 0$ and introduce the events

$$A_{n,\epsilon} = \{Z_{i'} > \epsilon a_n\} = \Big\{ \max_{1 \le i \le n-1} Z_i > \epsilon a_n \Big\}$$

and

$$B_{n,\epsilon} = \{Z_{i'} > \epsilon a_n \ \text{and} \ \exists\, l \neq 0, -i' \le l \le 1, \ \text{such that} \ Z_{i'+l} > \lambda \epsilon a_n\},$$

where $\lambda = c_0/(2c_1)$. Using the facts that (Z_i) is an i.i.d. sequence and $n\,\mathrm{P}(Z_1 > \epsilon a_n) \to 1/\epsilon$ as $n \to \infty$ (which follows from the regular variation property of Z_1) we obtain

$$\lim_{n \to \infty} \mathrm{P}(A_{n,\epsilon}) = \lim_{n \to \infty} [1 - (1 - \mathrm{P}(Z_1 > \epsilon a_n)^{n-1}] = 1 - e^{-1/\epsilon}. \tag{5.2}$$

Observe that

$$B_{n,\epsilon} \subseteq \bigcup_{i=1}^{n-1} \bigcup_{\substack{l = -(n-1) \\ l \neq 0}}^{1} \{Z_i > \epsilon a_n, Z_{i+l} > \lambda \epsilon a_n\}.$$

Then it holds that

$$\mathrm{P}(B_{n,\epsilon}) \le (n-1)n\, \mathrm{P}(Z_1 > \epsilon a_n)\, \mathrm{P}(Z_1 > \lambda \epsilon a_n) \to \frac{1}{\lambda \epsilon^2} \tag{5.3}$$

as $n \to \infty$.

On the event $A_{n,\epsilon} \setminus B_{n,\epsilon}$ one has $Z_{i'+l} \le \lambda \epsilon a_n$ for every $l \neq 0$, $-i' \le l \le 1$, so that

$$\bigvee_{j=1}^{i'-1} \frac{X_j}{a_n} = \max \bigg\{ c_0 \bigvee_{j=1}^{i'-1} \frac{Z_j}{a_n}, c_1 \bigvee_{j=0}^{i'-2} \frac{Z_j}{a_n} \bigg\} \le c_1 \lambda \epsilon = \frac{c_0 \epsilon}{2}$$

and

$$\bigvee_{j=1}^{i'} \frac{X_j}{a_n} = \max \bigg\{ \bigvee_{j=1}^{i'-1} \frac{X_j}{a_n}, \frac{X_{i'}}{a_n} \bigg\} \ge \frac{X_{i'}}{a_n} = \max \bigg\{ c_0 \frac{Z_{i'}}{a_n}, c_1 \frac{Z_{i'-1}}{a_n} \bigg\} \ge c_0 \frac{Z_{i'}}{a_n} \ge c_0 \epsilon.$$

Therefore

$$\Big| M_n\Big(\frac{i'}{n}\Big) - M_n\Big(\frac{i'-1}{n}\Big) \Big| = \bigg| \bigvee_{j=1}^{i'} \frac{X_j}{a_n} - \bigvee_{j=1}^{i'-1} \frac{X_j}{a_n} \bigg| \ge c_0 \epsilon - \frac{c_0 \epsilon}{2} = \frac{c_0 \epsilon}{2}. \tag{5.4}$$

On the event $A_{n,\epsilon} \setminus B_{n,\epsilon}$ one also have

$$\bigvee_{j=1}^{i'} \frac{X_j}{a_n} = \frac{X_{i'}}{a_n} = c_0 \frac{Z_{i'}}{a_n}$$

and

$$\bigvee_{j=1}^{i'+1} \frac{X_j}{a_n} \ge \frac{X_{i'+1}}{a_n} = \max \bigg\{ c_0 \frac{Z_{i'+1}}{a_n}, c_1 \frac{Z_{i'}}{a_n} \bigg\} \ge c_1 \frac{Z_{i'}}{a_n},$$

which imply

$$\begin{aligned}\Big|M_n\Big(\frac{i'+1}{n}\Big)-M_n\Big(\frac{i'}{n}\Big)\Big| &= \Big|\bigvee_{j=1}^{i'+1}\frac{X_j}{a_n}-\bigvee_{j=1}^{i'}\frac{X_j}{a_n}\Big| \\ &\geq (c_1-c_0)\frac{Z_{i'}}{a_n}=(c_1-c_0)\epsilon. \qquad (5.5)\end{aligned}$$

From (5.4) and (5.5) we obtain

$$\begin{aligned}\omega'_{2/n}(M_n(\,\cdot\,)) &\geq \min\Big\{\Big|M_n\Big(\frac{i'}{n}\Big)-M_n\Big(\frac{i'-1}{n}\Big)\Big|,\Big|M_n\Big(\frac{i'+1}{n}\Big)-M_n\Big(\frac{i'}{n}\Big)\Big|\Big\} \\ &\geq \epsilon\,\min\Big\{\frac{c_0}{2},c_1-c_0\Big\}>0\end{aligned}$$

on the event $A_{n,\epsilon}\setminus B_{n,\epsilon}$. Therefore, since $\omega'_\delta(\,\cdot\,)$ is nondecreasing in δ, it holds that

$$\begin{aligned}\liminf_{n\to\infty}\mathrm{P}(A_{n,\epsilon}\setminus B_{n,\epsilon}) &\leq \liminf_{n\to\infty}\mathrm{P}(\omega'_{2/n}(M_n(\,\cdot\,))\geq\epsilon\,\min\{c_0/2,c_1-c_0\}) \\ &\leq \lim_{\delta\to 0}\limsup_{n\to\infty}\mathrm{P}(\omega'_\delta(M_n(\,\cdot\,))\geq\epsilon\,\min\{c_0/2,c_1-c_0\}).\end{aligned}$$

Hence if we prove $\liminf_{n\to\infty}\mathrm{P}(A_{n,\epsilon}\setminus B_{n,\epsilon})>0$ for some $\epsilon>0$, then (5.1) will hold, and this will exclude the J_1 convergence. Since $x^2(1-e^{-1/x})$ tends to infinity as $x\to\infty$, we can find $\epsilon>0$ such that $\epsilon^2(1-e^{-1/\epsilon})>1/\lambda$, i.e.

$$1-e^{-1/\epsilon}>\frac{1}{\lambda\epsilon^2}.$$

Hence, taking into account relations (5.2) and (5.3) we obtain

$$\lim_{n\to\infty}\mathrm{P}(A_{n,\epsilon})>\limsup_{n\to\infty}\mathrm{P}(B_{n,\epsilon}),$$

and from this immediately follows

$$\liminf_{n\to\infty}\mathrm{P}(A_{n,\epsilon}\setminus B_{n,\epsilon})\geq\lim_{n\to\infty}\mathrm{P}(A_{n,\epsilon})-\limsup_{n\to\infty}\mathrm{P}(B_{n,\epsilon})>0.$$

Therefore the J_1 convergence does not hold.

Example 5.2. (squared GARCH process) We consider a stationary squared GARCH(1,1) process (X_n^2), where

$$X_n=\sigma_n Z_n,$$

with (Z_n) being a sequence of i.i.d. random variables with $\mathrm{E}(Z_1)=0$ and $\mathrm{var}(Z_1)=1$, and

$$\sigma_n^2=\alpha_0+(\alpha_1 Z_{n-1}^2+\beta_1)\sigma_{n-1}^2, \qquad (5.6)$$

with positive parameters α_1,β_1 and α_0. Assume that

$$-\infty\leq\mathrm{E}\ln(\alpha_1 Z_1^2+\beta_1)<0.$$

Then there exists a strictly stationary solution to the stochastic recurrence equation (5.6); see Goldie [12] and Mikosch and Stărică [21]. The process (X_n) is then strictly stationary too.

Assume that Z_1 is symmetric, has a positive Lebesgue density on $\mathbb{R}$ and there exists $\alpha>0$ such that

$$\mathrm{E}[(\alpha_1 Z_1^2+\beta_1)^\alpha]=1 \quad\text{and}\quad \mathrm{E}[(\alpha_1 Z_1^2+\beta_1)^\alpha\ln(\alpha_1 Z_1^2+\beta_1)]<\infty.$$

Then it is known that the processes (σ_n^2) and (X_n^2) are jointly regularly varying with index α and strongly mixing with geometric rate (see Basrak et al. [4]; Mikosch and Stărică [21]). Therefore the sequence (X_n^2) satisfies Condition 2.2. Condition 2.1 for the sequence (X_n^2) follows immediately from the results in Basrak et al. [4]. The extremal index of the sequence (X_n^2) is given by

$$\theta = \lim_{k\to\infty} \mathrm{E}\left(|Z_1|^{2\alpha} - \max_{j=2,\dots,k+1} \Big| Z_j^2 \prod_{i=1}^{j} (\alpha_1 Z_{i-1}^2 + \beta_1) \Big|^{\alpha} \right)_+ \Big/ \, \mathrm{E}\,|Z_1|^{2\alpha}$$

(see Mikosch and Stărică [21]).

Example 5.3. (ARMAX process) The ARMAX process is defined by

$$X_n = \max\{cX_{n-1}, Z_n\}, \qquad n \in \mathbb{Z}, \tag{5.7}$$

where $0 < c < 1$ and (Z_n) is a sequence of i.i.d. random variables with unit Fréchet distribution. According to Proposition 2.2 in Davis and Resnick [10] the unique stationary solution to (5.7) is given by

$$X_n = \bigvee_{i=0}^{\infty} c^i Z_{n-i}.$$

The process (X_n) is strongly mixing (see for example Ferreira and Ferreira [11], Proposition 3.1) and therefore Condition 2.2 holds. The joint regular variation property and Condition 2.1 for the process (X_n) can be obtained by an application of Theorem 2.3 and Theorem 2.4 in Meinguet [20]. The extremal index of the sequence (X_n) is given by $\theta = 1 - c$ (see Ancona-Navarrete and Tawn [2] and Ferreira and Ferreira [11]).

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