ON FUNCTIONAL WEAK CONVERGENCE FOR PARTIAL SUM PROCESSES

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ABSTRACT. For a strictly stationary sequence of regularly varying random variables we study functional weak convergence of partial sum processes in the space $D[0,1]$ with the J_1 topology. Under the strong mixing condition, we identify necessary and sufficient conditions for such convergence in terms of the corresponding extremal index. We also give conditions under which the regular variation property is a necessary condition for this functional convergence in the case of weak dependence.

1. INTRODUCTION

Let (X_n) be a strictly stationary sequence of real-valued random variables. If the sequence (X_n) is i.i.d. then it is well known (see for example Gnedenko and Kolmogorov [8], Rvačeva [16], Feller [7]) that there exist real sequences (a_n) and (b_n) such that

$$\frac{1}{a_n}\sum_{k=1}^{n}(X_k - b_n) \xrightarrow{d} S, \qquad (1.1)$$

for some non-degenerate α–stable random variable S with $\alpha \in (0,2)$ if and only if X_1 is *regularly varying with index* $\alpha \in (0,2)$, that is,

$$\mathrm{P}(|X_1| > x) = x^{-\alpha}L(x), \qquad (1.2)$$

where $L(\,\cdot\,)$ is a slowly varying function at ∞ and the tails are balanced: there exist $p, q \geq 0$ with $p + q = 1$ such that

$$\frac{\mathrm{P}(X_1 > x)}{\mathrm{P}(|X_1| > x)} \to p \qquad \text{and} \qquad \frac{\mathrm{P}(X_1 < -x)}{\mathrm{P}(|X_1| > x)} \to q, \qquad (1.3)$$

as $x \to \infty$. As α is less than 2, the variance of X_1 is infinite. The functional generalization of (1.1) has been studied extensively in probability literature. Define the partial sum processes

$$V_n(t) = \frac{1}{a_n}\sum_{k=1}^{\lfloor nt \rfloor} X_k - tb_n, \qquad t \in [0,1],$$

where the sequences (a_n) and (b_n) are chosen as

$$n\mathrm{P}(|X_1| > a_n) \to 1 \qquad \text{and} \qquad b_n = \frac{n}{a_n}\mathrm{E}\big(X_1\,1_{\{|X_1| \leq a_n\}}\big).$$

Here $\lfloor x \rfloor$ represents the integer part of the real number x. In functional limit theory one investigates the asymptotic behavior of the processes $V_n(\,\cdot\,)$ as $n \to \infty$. Since

the sample paths of $V_n(\,\cdot\,)$ are elements of the space $D[0,1]$ of all right-continuous real valued functions on $[0,1]$ with left limits, it is natural to consider the weak convergence of distributions of $V_n(\,\cdot\,)$ with the one of Skorohod topologies on $D[0,1]$ introduced in Skorohod [18].

A functional limit theorem for processes $V_n(\,\cdot\,)$ for infinite variance i.i.d. regularly varying sequences (X_n) was established by Skorohod [19]. Under some weak dependence conditions, weak convergence of partial sum processes in the Skorohod J_1 topology were obtained by Leadbetter and Rootzén [12] and Tyran-Kamińska [20]. They give a characterization of the J_1 convergence in terms of convergence of the corresponding point processes of jumps. Further in [20] are given sufficient conditions for such convergence when the stationary sequence is strongly mixing. One of them is a certain local dependence condition, which is implied by the local dependence condition D' of Davis [5]. It prevents clustering of large values of $|X_n|$, which allows the J_1 convergence to hold, since the J_1 topology is appropriate when extreme values do not cluster.

After recalling relevant notations and background in Section 2, in Section 3 we characterize the functional J_1 convergence of the partial sum process of a strictly stationary strongly mixing sequence (X_n) of regularly varying random variables in terms of the the extremal index of the sequence $(|X_n|)$, which is a standard tool in describing clustering of large values. When clustering of large values occurs J_1 convergence fails to hold, although convergence with respect to the weaker Skorohod M_1 topology might still hold. Recently Basrak et al. [2] gave sufficient conditions for functional limit theorem with the M_1 topology to hold for stationary, jointly regularly varying sequences for which all extremes within each cluster of high-threshold excesses have the same sign.

The regular variation property is a necessary condition for the J_1 convergence of the partial sum process in the i.i.d. case (see for example Corollary 7.1 in Resnick [15]). In Section 4 we extend this result to the weak dependent case when clustering of large values do not occur.

2. Preliminaries

In this section we introduce some basic tools and notions to be used throughout the paper.

2.1. **Regular Variation.** Let $\mathbb{E} = \overline{\mathbb{R}} \setminus \{0\}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. The space $\mathbb{E}$ is equipped with the topology by which the Borel σ–algebras $\mathcal{B}(\mathbb{E})$ and $\mathcal{B}(\mathbb{R})$ coincide on $\mathbb{R} \setminus \{0\}$. A set $B \subseteq \mathbb{E}$ is relatively compact if it is bounded away from origin, that is, if there exists $u > 0$ such that $B \subseteq \mathbb{E} \setminus [-u, u]$. Let $M_+(\mathbb{E})$ be the class of all Radon measures on $\mathbb{E}$, i.e. all nonnegative measures that are finite on relatively compact subsets of $\mathbb{E}$. A useful topology for $M_+(\mathbb{E})$ is the vague topology which renders $M_+(\mathbb{E})$ a complete separable metric space. If $\mu_n \in M_+(\mathbb{E})$, $n \geq 0$, then μ_n converges vaguely to μ_0 (written $\mu_n \xrightarrow{v} \mu_0$) if $\int f\,d\mu_n \to \int f\,d\mu_0$ for all $f \in C_K^+(\mathbb{E})$, where $C_K^+(\mathbb{E})$ denotes the class of all nonnegative continuous real functions on $\mathbb{E}$ with compact support.

Regular variation can be expressed in terms of vague convergence of measures on $\mathbb{E}$. Relation (1.2) together with (1.3) are equivalent to

$$n\mathrm{P}(a_n^{-1}X_1 \in \,\cdot\,) \xrightarrow{v} \mu(\,\cdot\,) \qquad \text{as } n \to \infty,$$

the Radon measure μ on $\mathbb{E}$ being given by

$$\mu(dx) = \big(p\alpha x^{-\alpha-1}1_{(0,\infty)}(x) + q\alpha(-x)^{-\alpha-1}1_{(-\infty,0)}(x)\big)dx, \qquad (2.1)$$

where p and q are as in (1.3).

Using standard regular variation arguments it can be shown that for every $\lambda > 0$ it holds that

$$\frac{a_{\lfloor \lambda n \rfloor}}{a_n} \to \lambda^{1/\alpha} \qquad \text{as } n \to \infty.$$

Therefore a_n can be represented as

$$a_n = n^{1/\alpha}L'(n),$$

where $L'(\,\cdot\,)$ is a slowly varying function at ∞.

2.2. **Skorohod J_1 and M_1 topologies.** Since the stochastic processes that we consider in this paper have discontinuities, for the function space of sample paths of these stochastic processes we take the space $D[0,1]$ of all right-continuous real valued functions on $[0,1]$ with left limits. Usually the space $D[0,1]$ is endowed with the Skorohod J_1 topology, which is appropriate when clustering of large values do not occur.

The metric d_{J_1} that generates the J_1 topology on $D[0,1]$ is defined in the following way. Let Δ be the set of strictly increasing continuous functions $\lambda\colon [0,1] \to [0,1]$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$, and let $e \in \Delta$ be the identity map on $[0,1]$, i.e. $e(t) = t$ for all $t \in [0,1]$. For $x, y \in D[0,1]$ define

$$d_{J_1}(x,y) = \inf\{\|x \circ \lambda - y\|_{[0,1]} \vee \|\lambda - e\|_{[0,1]} : \lambda \in \Delta\},$$

where $\|x\|_{[0,1]} = \sup\{|x(t)| : t \in [0,1]\}$ and $a \vee b = \max\{a, b\}$. Then d_{J_1} is a metric on $D[0,1]$ and is called the Skorohod J_1 metric.

When stochastic processes exhibit rapid successions of jumps within temporal clusters of large values, collapsing in the limit to a single jump, the J_1 topology become inappropriate since the J_1 convergence fails to hold. This difficulty can be overcame by using a weaker topology in which the functional convergence may still hold. i.e. the Skorohod M_1 topology.

The M_1 metric d_{M_1} that generates the M_1 topology is defined using the completed graphs. For $x \in D[0,1]$ the *completed graph* of x is the set

$$\Gamma_x = \{(t,z) \in [0,1] \times \mathbb{R} : z = \lambda x(t-) + (1-\lambda)x(t) \text{ for some } \lambda \in [0,1]\},$$

where $x(t-)$ is the left limit of x at t. Thus the completed graph of x besides the points of the graph $\{(t,x(t)) : t \in [0,1]\}$ contains also the vertical line segments joining $(t,x(t))$ and $(t,x(t-))$ for all discontinuity points t of x. We define an *order* on the graph Γ_x by saying that $(t_1,z_1) \leqslant (t_2,z_2)$ if either (i) $t_1 < t_2$ or (ii) $t_1 = t_2$ and $|x(t_1-) - z_1| \leqslant |x(t_2-) - z_2|$. A *parametric representation* of the completed graph Γ_x is a continuous nondecreasing function (r,u) mapping $[0,1]$ onto Γ_x, with r being the time component and u being the spatial component. Denote by $\Pi(x)$ the set of parametric representations of the graph Γ_x. For $x_1, x_2 \in D[0,1]$ define

$$d_{M_1}(x_1,x_2) = \inf\{\|r_1 - r_2\|_{[0,1]} \vee \|u_1 - u_2\|_{[0,1]} : (r_i, u_i) \in \Pi(x_i), i = 1, 2\}.$$

This definition introduces d_{M_1} as a metric on $D[0,1]$. The induced topology is called the Skorohod M_1 topology.

The J_1 and M_1 metrics are related by the following inequality

$$d_{M_1}(x,y) \leq d_{J_1}(x,y), \qquad x, y \in D[0,1]$$

(see for instance Theorem 6.3.2 in Whitt [21]).

Define the maximum (absolute) jump functional $T\colon D[0,1] \to \mathbb{R}$ by

$$T(x) = \sup\{|x(t) - x(t-)| : t \in [0,1]\}, \qquad x \in D[0,1]. \tag{2.2}$$

The supremum in (2.2) is always attained, because for any $\epsilon > 0$ a function $x \in D[0,1]$ has finitely many jumps of magnitude greater than ϵ (cf. Whitt [21], page 98). If $d_{M_1}(x_n, x) \to 0$, by Lemma 4.1 of Pang and Whitt [14] (cf. Whitt [21], Corollary 12.11.3) it follows that

$$\limsup_{n\to\infty} T(x_n) \le T(x). \tag{2.3}$$

Lemma 2.1. *The functional $T\colon D[0,1] \to \mathbb{R}$ is continuous on $D[0,1]$ when $D[0,1]$ is endowed with the Skorohod J_1 topology.*

Proof. Take an arbitrary $x \in D[0,1]$ and suppose that $d_{J_1}(x_n, x) \to 0$ in $D[0,1]$. Since the J_1 convergence implies the M_1 convergence, by relation (2.3) we have $\limsup_{n\to\infty} T(x_n) \le T(x)$.

From the definition of the functional T, we know that there exists some $t_0 \in (0,1]$ such that $|x(t_0) - x(t_0-)| = T(x)$. From the definition of the J_1 metric, there exists a sequence (t_n) in $[0,1]$ such that $t_n \to t_0$, $x_n(t_n) \to x(t_0)$ and $x_n(t_n-) \to x(t_0-)$ as $n \to \infty$ (it suffices to let $t_n = \lambda_n(t_0)$ where $\|\lambda_n - e\|_{[0,1]} \to 0$ and $\|x_n \circ \lambda_n - x\|_{[0,1]} \to 0$). Hence

$$|x_n(t_n) - x_n(t_n-)| \to |x(t_0) - x(t_0-)| = T(x) \qquad \text{as } n \to \infty.$$

Nothing that

$$T(x_n) = \sup\{|x_n(t) - x_n(t-)| : t \in [0,1]\} \ge |x_n(t_n) - x_n(t_n-)|,$$

we obtain $\liminf_{n\to\infty} T(x_n) \ge T(x)$. Therefore $\lim_{n\to\infty} T(x_n) = T(x)$, and we conclude that T is continuous at x. $\qquad\square$

Remark 2.2. Similar results as in Lemma 2.1 hold for the maximum positive and negative jump functionals $T^+, T^-\colon D[0,1] \to \mathbb{R}$ defined by

$$T^+(x) \;\; = \;\; \sup\{|x(t) - x(t-)|1_{\{x(t)-x(t-)>0\}} : t \in [0,1]\},$$

$$T^-(x) \;\; = \;\; \sup\{|x(t) - x(t-)|1_{\{x(t)-x(t-)<0\}} : t \in [0,1]\},$$

for $x \in D[0,1]$. Precisely, the functionals T^+ and T^- are continuous on $D[0,1]$ when $D[0,1]$ is endowed with the Skorohod J_1 topology. This can be proven using a slight modification of the proof of Lemma 4.1 in Pang and Whitt [14] and the procedure used in the proof of Lemma 2.1 (we omit the details here).

For more discussion about the J_1 and M_1 topologies we refer to Resnick [15], section 3.3.4 and Whitt [21], sections 12.3–12.5.

2.3. **Weak dependence.** A strictly stationary sequence (ξ_n) has extremal index θ if for every $\tau > 0$ there exists a sequence of real numbers (u_n) such that

$$\lim_{n\to\infty} n\mathrm{P}(\xi_1 > u_n) \to \tau \qquad \text{and} \qquad \lim_{n\to\infty} \mathrm{P}\left(\max_{1\le i\le n} \xi_i \le u_n\right) \to e^{-\theta\tau}. \tag{2.4}$$

It holds that $\theta \in [0,1]$. In particular, if the ξ_n are i.i.d. then (2.4) can hold only for $\theta = 1$. Dependent random variables can also have extremal index equal to 1. For this it suffices that they satisfy the extreme mixing conditions $D(u_n)$ and $D'(u_n)$ introduced by Leadbetter [10], [11]. The extremal index can be interpreted as the

reciprocal mean cluster size of large exceedances (cf. Hsing et al. [9]). When $\theta < 1$ clustering of extreme values occurs. If the sequence (ξ_n) is strongly mixing and the ξ_n's are regularly varying then for θ to be the extremal index of (ξ_n) it suffices that (2.4) holds for some $\tau > 0$ (cf. Leadbetter and Rootzén [12], page 439).

In order to restrict the dependence in the sequence (X_n) we will use the strong mixing condition. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space. For any σ-field $\mathcal{A} \subset \mathcal{F}$, let $L_2(\mathcal{A})$ denote the space of square-integrable, $\mathcal{A}$-measurable, real-valued random variables. For any two σ-fields $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ define

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathrm{P}(A \cap B) - \mathrm{P}(A)\mathrm{P}(B)| : A \in \mathcal{A}, B \in \mathcal{B}\}$$

For a sequence (X_n) of random variables on $(\Omega, \mathcal{F}, \mathrm{P})$, we define $\mathcal{F}_k^l = \sigma(\{X_i : k \le i \le l\})$. Then we say the sequence (X_n) is α–mixing (or strongly mixing) if

$$\alpha_n = \sup_{j \ge 1} \alpha(\mathcal{F}_1^j, \mathcal{F}_{j+n}^\infty) \to 0 \qquad \text{as } n \to \infty.$$

Let (u_n) be a sequence of real numbers and (q_n) any sequence of positive integers with $q_n \to \infty$ as $n \to \infty$ and $q_n = o(n)$. O'Brien [13] showed that if the sequence (X_n) is strongly mixing and there exists a sequence (p_n) of positive integers such that $p_n = o(n)$, $n\alpha_{q_n} = o(p_n)$, $q_n = o(p_n)$, and either $\liminf [\mathrm{P}(X_1 \le u_n)]^n > 0$ or $\liminf \mathrm{P}(M_{2,p_n} \le u_n | X_1 > u_n) > 0$, then

$$\mathrm{P}(M_n \le u_n) - [\mathrm{P}(X_1 \le u_n)]^{n\mathrm{P}(M_{2,p_n} \le u_n | X_1 > u_n)} \to 0 \qquad \text{as } n \to \infty, \qquad (2.5)$$

where $M_{i,j} = \max\{X_k : k = i, \dots, j\}$.

3. Limit theorem with J_1 convergence

Let (X_n) be a strongly mixing and strictly stationary sequence of regularly varying random variables with index $\alpha \in (0, 2)$. Let (a_n) be a sequence of positive real numbers such that

$$n\mathrm{P}(|X_1| > a_n) \to 1 \qquad \text{as } n \to \infty. \qquad (3.1)$$

Tyran-Kamińska [20] showed that under a certain "vanishing small values" condition when $\alpha \in [1, 2)$ (see Condition 3.2 below), the partial sum process

$$V_n(t) = \sum_{k=1}^{[nt]} \frac{X_k}{a_n} - nt\,\mathrm{E}\left(\frac{X_1}{a_n} \mathbf{1}_{\left\{\frac{|X_1|}{a_n} \le 1\right\}}\right), \quad t \in [0, 1],$$

satisfies a nonstandard functional limit theorem in the space $D[0, 1]$ equipped with the Skorohod J_1 topology, with a Lévy α–stable process as a limit if and only if the following local dependence condition holds:

Condition 3.1. For any $x > 0$ there exist sequences of integers $p_n = p_n(x)$, $q_n = q_n(x) \to \infty$ such that

$$p_n = o(n), \quad n\alpha_{q_n} = o(p_n), \quad q_n = o(p_n) \qquad \text{as } n \to \infty,$$

and

$$\lim_{n \to \infty} \mathrm{P}\left(\max_{2 \le i \le p_n} |X_i| > xa_n \,\Big|\, |X_1| > xa_n\right) = 0. \qquad (3.2)$$

Here (α_n) is the sequence of α–mixing coefficients of (X_n).

We will show a similar result, but with a certain condition involving the extremal index of the sequence $(|X_n|)$ instead of Condition 3.1. Roughly speaking Condition 3.1 prevents clustering of large values, which is well known to occur when the extremal index of the sequence is equal to 1. Hence it is expected that the functional J_1 convergence of the partial sum process holds if and only if the sequence $(|X_n|)$ has extremal index equal to 1, which we formally prove in the following theorem.

Recall that the distribution of a Lévy process $W(\cdot)$ is characterized by its characteristic triplet, i.e. the characteristic triplet of the infinitely divisible distribution of $W(1)$. The characteristic function of $W(1)$ and the characteristic triplet (a, μ, b) are related in the following way:

$$\mathrm{E}[e^{izW(1)}] = \exp\left(-\frac{1}{2}az^2 + ibz + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx1_{[-1,1]}(x)\right)\mu(dx)\right)$$

for $z \in \mathbb{R}$; here $a \geq 0$, $b \in \mathbb{R}$ are constants, and μ is a measure on $\mathbb{R}$ satisfying

$$\mu(\{0\}) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}} (|x|^2 \wedge 1)\mu(dx) < \infty,$$

that is, μ is a Lévy measure. For a textbook treatment of Lévy processes we refer to Bertoin [3] and Sato [17].

In this section we identify some necessary and sufficient conditions for the J_1 convergence of partial sum processes $V_n(\cdot)$ to a Lévy stable process. In case $\alpha \in [1, 2)$, we will need to assume that the contribution of the smaller increments of the partial sum process is close to its expectation.

Condition 3.2. For all $\delta > 0$,

$$\limsup_{u \downarrow 0} \limsup_{n \to \infty} \mathrm{P}\left[\max_{0 \leq k \leq n} \left|\sum_{i=1}^{k} \left(\frac{X_i}{a_n}1_{\left\{\frac{|X_i|}{a_n} \leq u\right\}} - \mathrm{E}\left(\frac{X_i}{a_n}1_{\left\{\frac{|X_i|}{a_n} \leq u\right\}}\right)\right)\right| > \delta\right] = 0.$$

Theorem 3.3. *Let (X_n) be a strictly stationary sequence of regularly varying random variables with index $\alpha \in (0, 2)$. Assume the sequence (X_n) is strongly mixing, and if $1 \leq \alpha < 2$, also suppose that Condition 3.2 holds. Then*

$$V_n \xrightarrow{d} V, \qquad n \to \infty,$$

in $D[0, 1]$ endowed with the J_1 topology, where $V(\cdot)$ is an α–stable Lévy process with characteristic triplet $(0, \mu, 0)$ and μ as in (2.1), if and only if the sequence $(|X_n|)$ has extremal index $\theta = 1$.

Proof. First assume the sequence $(|X_n|)$ has extremal index $\theta = 1$. Let (q_n) be any sequence of positive integers such that $q_n \to \infty$ and $q_n = o(n)$. Fix an arbitrary $x > 0$ and put $p_n = \max\{\lfloor n\sqrt{\alpha_{q_n}}\rfloor, \lfloor \sqrt{nq_n}\rfloor + 1\}$, where (α_n) is the sequence of α–mixing coefficients of (X_n). Then it can easily be seen that $p_n = o(n)$, $n\alpha_{q_n} = o(p_n)$ and $q_n = o(p_n)$. Since by a standard regular variation argument

$$\lim_{n \to \infty} [\mathrm{P}(|X_1| \leq xa_n)]^n = \lim_{n \to \infty} \left[1 - \frac{n\mathrm{P}(|X_1| > xa_n)}{n}\right]^n = e^{-x^{-\alpha}}, \qquad (3.3)$$

i.e. $\liminf_{n \to \infty}[\mathrm{P}(|X_1| \leq xa_n)]^n = e^{-x^{-\alpha}} > 0$, from relation (2.5) we obtain that, as $n \to \infty$,

$$\mathrm{P}\left(\max_{1 \leq i \leq n} |X_i| \leq xa_n\right) - [\mathrm{P}(|X_1| \leq xa_n)]^{t_n} \to 0, \qquad (3.4)$$

where $t_n = n\mathrm{P}(\max_{2 \le i \le p_n} |X_i| \le xa_n \,|\, |X_1| > xa_n)$.

Let $(\widehat{X}_n)$ be the associated independent sequence of (X_n), i.e. $(\widehat{X}_n)$ is an i.i.d. sequence with $\widehat{X}_1 \overset{d}{=} X_1$. Then by Theorem 2.2.1 in Leadbetter and Rootzén [12]

$$\mathrm{P}\left(\max_{1 \le i \le n} |X_i| \le xa_n \right) \to G^\theta(x) \qquad \text{as } n \to \infty, \tag{3.5}$$

where

$$G(x) = \lim_{n \to \infty} \mathrm{P}\left(\max_{1 \le i \le n} |\widehat{X}_i| \le xa_n \right) = \lim_{n \to \infty} [\mathrm{P}(|\widehat{X}_1| \le xa_n)]^n = e^{-x^{-\alpha}}.$$

Since $\theta = 1$, from (3.5) we obtain

$$\mathrm{P}\left(\max_{1 \le i \le n} |X_i| \le xa_n \right) \to e^{-x^{-\alpha}} \qquad \text{as } n \to \infty. \tag{3.6}$$

Therefore, from (3.4) and (3.6) we obtain, as $n \to \infty$,

$$\mathrm{P}\left(\max_{2 \le i \le p_n} |X_i| \le xa_n \,\bigg|\, |X_1| > xa_n \right) \cdot \ln[\mathrm{P}(|X_1| \le xa_n)]^n \to -x^{-\alpha},$$

and taking into account relation (3.3) it follows that

$$\mathrm{P}\left(\max_{2 \le i \le p_n} |X_i| > xa_n \,\bigg|\, |X_1| > xa_n \right) \to 0 \qquad \text{as } n \to \infty.$$

Therefore (3.2) holds and an application of Theorem 1.1 in Tyran-Kamińska [20] yields that $V_n \overset{d}{\to} V$ in $D[0,1]$ with the Skorohod J_1 topology.

Now assume $V_n \overset{d}{\to} V$ in $D[0,1]$ with the J_1 topology. Since by Lemma 2.1 the functional T is continuous, by the continuous mapping theorem we obtain

$$T(V_n) = \frac{1}{a_n} \bigvee_{i=1}^n |X_i| \overset{d}{\to} T(V) \qquad \text{as } n \to \infty. \tag{3.7}$$

Following the Lévy-Ito representation of Lévy processes, $V(\cdot)$ can be represented as

$$V(t) = \sum_{t_k \le t} j_k \mathbf{1}_{\{|j_k| > 1\}} + \lim_{\epsilon \downarrow 0} \left(\sum_{t_k \le t} j_k \mathbf{1}_{\{\epsilon < |j_k| \le 1\}} - t \int_{\epsilon < |x| \le 1} x \, \mu(dx) \right),$$

with $N = \sum_k \delta_{(t_k, j_k)}$ being a Poisson process with mean measure $\lambda \times \mu$, where λ is the Lebesgue measure (see Resnick [15], page 150). Since $T(V) = \sup\{|j_k| : t_k \le 1\}$, we have for every $x > 0$,

$$\mathrm{P}(T(V) \le x) = \mathrm{P}(N([0,1] \times \mathbb{E}_x) = 0) = e^{-x^{-\alpha}}.$$

Hence (3.7) yields

$$\lim_{n \to \infty} \mathrm{P}\left(\frac{1}{a_n} \bigvee_{i=1}^n |X_i| \le x \right) \to e^{-x^{-\alpha}}.$$

Taking $x = 1$, we obtain

$$\lim_{n \to \infty} \mathrm{P}\left(\bigvee_{i=1}^n |X_i| \le a_n \right) \to e^{-1},$$

and from this, taking into account relation (3.1) we conclude that $(|X_n|)$ has extremal index equal to 1. $\qquad \square$

Remark 3.4. If random variables X_n are regularly varying with index $\alpha \in (0,1)$, the centering function in the definition of the process $V_n(\cdot)$ can be removed and this removing affects the characteristic triplet of the limiting process in the way we describe here.

By Karamata's theorem, as $n \to \infty$,

$$n\mathrm{E}\left(\frac{X_1}{a_n} 1_{\left\{ \frac{|X_1|}{a_n} \leq 1 \right\}} \right) \to (p-q)\frac{\alpha}{1-\alpha}, \tag{3.8}$$

where p and q are as in (2.1). Put $b = (p-q)\alpha/(1-\alpha)$ and define the function $x_0 \colon [0,1] \to \mathbb{R}$ by $x_0(t) = bt$. The function x_0 is continuous, and hence it belongs to $D[0,1]$. Further, by standard arguments one can show that the function $h \colon D[0,1] \to D[0,1]$ defined by $h(x) = x + x_0$ is continuous (with respect to the J_1 topology on $D[0,1]$). Hence by the continuous mapping theorem from $V_n \xrightarrow{d} V$ we obtain $h(V_n) \xrightarrow{d} h(V)$.

Since the J_1 metric on $D[0,1]$ is bounded above by the uniform metric on $D[0,1]$, for every $\delta > 0$ it holds

$$\mathrm{P}\left[d_{J_1}\left(h(V_n(\cdot)), \sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} \right) > \delta \right] \leq \mathrm{P}\left(\sup_{t \in [0,1]} \left| nt\,\mathrm{E}\left(\frac{X_1}{a_n} 1_{\left\{ \frac{|X_1|}{a_n} \leq 1 \right\}} \right) - bt \right| > \delta \right)$$

$$= \mathrm{P}\left(\left| n\,\mathrm{E}\left(\frac{X_1}{a_n} 1_{\left\{ \frac{|X_1|}{a_n} \leq 1 \right\}} \right) - b \right| > \delta \right).$$

Hence (3.8) yields

$$\lim_{n \to \infty} \mathrm{P}\left[d_{J_1}\left(h(V_n(\cdot)), \sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} \right) > \delta \right] = 0,$$

and an application of Slutsky's theorem (cf. Theorem 3.4 in Resnick [15]) leads to

$$\sum_{k=1}^{\lfloor n \cdot \rfloor} \frac{X_k}{a_n} \xrightarrow{d} h(V(\cdot)) = V(\cdot) + (\cdot)b$$

in $D[0,1]$ endowed with the J_1 topology. The characteristic triplet of the limiting process is therefore $(0, \mu, b)$.

Remark 3.5. The J_1 convergence in Theorem 3.3 fails to hold when the extremal index $\theta < 1$. For example, the extremal index of the moving average process

$$X_n = Y_n + Y_{n+1}, \qquad n \in \mathbb{Z},$$

where (Y_n) is an i.i.d. sequence of regularly varying random variables, is equal to $1/2$ (cf. Leadbetter and Rootzén [12] or Embrechts et al. [6], page 415). By Theorem 1 of Avram and Taqqu [1], the J_1 convergence does not hold for this process.

In this case the J_1 topology is inappropriate as the partial sum process may exhibit rapid successions of jumps within temporal clusters of large values, collapsing in the limit to a single jump. In other words the J_1 convergence could hold only if extreme values do not cluster, i.e. when $\theta = 1$.

When $\theta < 1$, the functional convergence may still hold in the weaker M_1 topology. Sufficient conditions for such convergence were obtained by Basrak et al. [2].

4. Necessity of the regular variation condition

The theorem below gives a certain converse of Theorem 3.3. It gives conditions under which the regular variation property of X_n's is a necessary condition for the functional J_1 convergence of partial sum processes $V_n(\cdot)$ to a Lévy stable process. This can be viewed as a generalization of the corresponding result for i.i.d. random variables (cf. Corollary 7.1 in Resnick [15]). In the dependent case some restrictions on the sequence (X_n) are necessary, since in general the functional convergence may hold although the random variables are not regularly varying. This can be seen in the following example.

Example 4.1. Let (X_i) be a sequence of i.i.d. Pareto random variables with the tail function $P(X_i > x) = 1_{(-\infty,1)}(x) + x^{-1/2}1_{[1,\infty)}(x)$, $x \in \mathbb{R}$. Hence X_i is regularly varying with index $\alpha = 1/2$. Assume Z is a random variable with the tail function $P(Z > x) = 1_{(-\infty,e)}(x) + (\log x)^{-1}1_{[e,\infty)}(x)$, and independent of the sequence (X_i). Here $\log x$ represents the natural logarithm of the positive real number x. Since the random variables X_i are i.i.d. and regularly varying it is well known that

$$\sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} \xrightarrow{d} \widetilde{V}(\cdot), \qquad n \to \infty, \tag{4.1}$$

in $D[0,1]$ with the J_1 topology, where the limiting process $\widetilde{V}(\cdot)$ is an α–stable Lévy process with characteristic triplet $(0, \mu, (p-q)\alpha/(1-\alpha))$ with μ as in (2.1) (see for instance Theorem 7.1 and Corollary 7.1 in Resnick [15]). This follows also from Theorem 3.3 since the i.i.d. sequence (X_i) has extremal index $\theta = 1$.

Using again the fact that the J_1 metric on $D[0,1]$ is bounded above by the uniform metric on $D[0,1]$, for every $\delta > 0$ we obtain

$$P\left[d_{J_1}\left(\sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n}, \sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} + \lfloor n \cdot \rfloor \frac{Z}{a_n}\right) > \delta\right] \leq P\left(\sup_{t \in [0,1]} \left|\lfloor nt \rfloor \frac{Z}{a_n}\right| > \delta\right)$$

$$= P\left(\frac{n}{a_n}|Z| > \delta\right).$$

Recall that a_n can be represented as $a_n = n^{1/\alpha}L'(n)$, with $L'(\cdot)$ being a slowly varying function at ∞. Using this and Proposition 1.3.6 in Bingham et al. [4] we obtain

$$\frac{n}{a_n} = \frac{1}{n^{1/\alpha-1}L'(n)} \to 0 \qquad \text{as } n \to \infty.$$

This implies $P(n|Z|/a_n > \delta) \to 0$ as $n \to \infty$, and hence

$$\lim_{n \to \infty} P\left[d_{J_1}\left(\sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n}, \sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} + \lfloor n \cdot \rfloor \frac{Z}{a_n}\right) > \delta\right] = 0.$$

By Slutsky's theorem, as $n \to \infty$,

$$\sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i + Z}{a_n} = \sum_{i=1}^{\lfloor n \cdot \rfloor} \frac{X_i}{a_n} + \lfloor n \cdot \rfloor \frac{Z}{a_n} \xrightarrow{d} \widetilde{V}(\cdot),$$

in $D[0,1]$ with the J_1 topology. Therefore, the functional J_1 convergence holds for the sequence (Y_i), where $Y_i = X_i + Z$.

Let show now that random variables Y_i are not regularly varying. For large $x > 0$ it holds that $\mathrm{P}(Y_1 > x) \geq \mathrm{P}(Z > x)$. If we assume Y_1 is regularly varying, then it would hold $\mathrm{P}(Y_1 > x) = x^{-\beta} L(x)$ for some $\beta > 0$ and some slowly varying function $L(\,\cdot\,)$. Hence $x^{-\beta} L(x) \geq (\log x)^{-1}$, i.e.

$$x^{-\beta/2} L(x) \geq x^{\beta/2} (\log x)^{-1}.$$

Letting $x \to \infty$, we obtain 0 on the left hand side of this inequality (by Proposition 1.3.6 in Bingham et al. [4]) and ∞ on the right hand side, which is a contradiction. Therefore, Y_1 is not regularly varying.

Theorem 4.2. *Let (X_n) be a strictly stationary sequence of random variables. Suppose that $(|X_n|)$ has extremal index $\theta = 1$ and that the sequences (X_n^+) and (X_n^-) have positive extremal indexes. If $V_n \xrightarrow{d} V$ in $D[0,1]$ endowed with the J_1 topology, where $V(\,\cdot\,)$ is an α–stable Lévy process with characteristic triplet $(0, \mu, 0)$,*

$$\mu(dx) = \bigl(p\alpha x^{-\alpha-1} 1_{(0,\infty)}(x) + q\alpha(-x)^{-\alpha-1} 1_{(-\infty,0)}(x)\bigr) dx,$$

$(p, q \geq 0,\ p + q = 1)$ and $\alpha \in (0, 2)$, then

$$n\mathrm{P}(a_n^{-1} X_1 \in \,\cdot\,) \xrightarrow{v} \mu(\,\cdot\,) \qquad as\ n \to \infty,$$

i.e. X_1 is regularly varying with index α.

Proof. Let $x > 0$ be arbitrary. As in the second part of the proof of Theorem 3.3, applying the functional $T \colon D[0,1] \to \mathbb{R}$ to the convergence $V_n \xrightarrow{d} V$ we obtain

$$\lim_{n\to\infty} \mathrm{P}\biggl(\frac{1}{a_n} \bigvee_{i=1}^{n} |X_i| \leq x\biggr) \to e^{-x^{-\alpha}}. \tag{4.2}$$

In the same way, applying the functional T^+ to the convergence $V_n \xrightarrow{d} V$ yields

$$\lim_{n\to\infty} \mathrm{P}\biggl(\frac{1}{a_n} \bigvee_{i=1}^{n} X_i^+ \leq x\biggr) \to e^{-px^{-\alpha}}. \tag{4.3}$$

Here we used the fact that

$$\mathrm{P}(T^+(V) \leq x) = \mathrm{P}(N([0,1] \times (x,\infty]) = 0) = e^{-px^{-\alpha}},$$

with N being a Poisson process with mean measure $\lambda \times \mu$ (as described in the proof of Theorem 3.3). Using Theorem 2.2.1 in Leadbetter and Rootzén [12], from (4.3) we obtain

$$\lim_{n\to\infty} \mathrm{P}\biggl(\frac{1}{a_n} \bigvee_{i=1}^{n} \widehat{X}_i^+ \leq x\biggr) \to e^{-\frac{p}{\theta_1} x^{-\alpha}},$$

where θ_1 is the extremal index of (X_n^+) and $(\widehat{X}_n)$ is the associated independent sequence of (X_n). Now by Lemma 1.2.2 in Leadbetter and Rootzén [12] (cf. also Proposition 7.1 in Resnick [15]) it holds that

$$n\mathrm{P}(X_1^+ > xa_n) \to \frac{p}{\theta_1} x^{-\alpha} \qquad as\ n \to \infty. \tag{4.4}$$

Repeating the same procedure for the functional T^- we obtain

$$n\mathrm{P}(X_1^- > xa_n) \to \frac{q}{\theta_2} x^{-\alpha} \qquad as\ n \to \infty, \tag{4.5}$$

where θ_2 is the extremal index of (X_n^-). In the same manner from (4.2) we get

$$n\mathrm{P}(|X_1| > xa_n) \to \frac{1}{\theta}x^{-\alpha} = x^{-\alpha} \qquad \text{as } n \to \infty.$$

On the other hand,

$$n\mathrm{P}(|X_1| > xa_n) = n\mathrm{P}(X_1^+ > xa_n) + n\mathrm{P}(X_1^- > xa_n) \to \left(\frac{p}{\theta_1} + \frac{q}{\theta_2}\right)x^{-\alpha} \qquad \text{as } n \to \infty,$$

and hence we conclude that

$$\frac{p}{\theta_1} + \frac{q}{\theta_2} = 1. \tag{4.6}$$

Since $\theta_1, \theta_2 \in (0, 1]$ and $p + q = 1$, from relation (4.6) we obtain $\theta_1 = \theta_2 = 1$. Now, from (4.4) and (4.5) it follows (cf. Lemma 6.1 in Resnick [15])

$$n\mathrm{P}(a_n^{-1}X_1 \in \cdot) \xrightarrow{v} \mu(\cdot) \qquad \text{as } n \to \infty.$$

$\square$

Remark 4.3. If the random variables X_n that appear in Theorem 4.2 are positive, then the conditions on extremal indexes of $(|X_n|)$, (X_n^+) and (X_n^-) reduce to a single condition, i.e. that (X_n) has extremal index equal to 1. The same holds if the X_n are negative.

Remark 4.4. From the functional M_1 convergence of the partial sum process, using arguments as in the proof of Theorem 4.2, we can not obtain the regular variation property for random variables X_n. This is due to the fact that the maximum jump functional T is not continuous on $D[0, 1]$ with respect to the Skorohod M_1 topology (see the comment after Lemma 4.1 in Pang and Whitt [14]).

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