Functional limit theorems for weakly dependent regularly varying time series

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Introduction

Functional limit theorems for sequences of independent and identically distributed random variables have been known for quite some time. These theorems can be divided into two main groups depending on whether the second moments of the underlying random variables are finite or infinite. The first step toward generalization of these results is to replace independence by some weak dependence property. Such questions will be studied in this thesis.

Let $(X_n)_{n\geqslant 1}$ be a strictly stationary sequence of random variables and let $S_n = X_1 + \cdots + X_n,\ n \geqslant 1$, denote its corresponding sequence of partial sums. The main goal of this thesis is to investigate the asymptotic distributional behavior of the $D[0,1]$ valued process

$$V_n(t) = a_n^{-1}(S_{\lfloor nt\rfloor} - \lfloor nt\rfloor b_n), \qquad t \in [0,1],$$

under the properties of weak dependence and regular variation with index $\alpha \in (0,2)$, where (a_n) is a sequence of positive real numbers such that

$$n\,\mathrm{P}(|X_1| > a_n) \to 1,$$

as $n \to \infty$, and

$$b_n = \mathrm{E}\big(X_1\, 1_{\{|X_1|\leqslant a_n\}}\big).$$

Here, $\lfloor x\rfloor$ represents the integer part of the real number x and $D[0,1]$ is the space of real-valued right continuous functions on $[0,1]$ with left limits.

Recall that if the sequence (X_n) is i.i.d. and if there exist real sequences (a_n) and (b_n) and a non-degenerate random variable S such that as $n \to \infty$,

$$\frac{S_n - b_n}{a_n} \xrightarrow{d} S, \tag{0.1}$$

then S is necessarily an α–stable random variable, i.e. the law of X_1 belongs to the domain of attraction of S. Classical references in the i.i.d. case are the books by Feller [32] and Petrov [56], while in LePage et al. [46] we can find an elegant probabilistic proof of sufficiency and a nice representation of the limiting distribution.

Weakly dependent sequences can exhibit very similar behavior. In [22], Davis proved that if a sequence (X_n) of regularly varying random variables with tail index $\alpha \in (0, 2)$ satisfies a strengthened version of Leadbetter's D and D' conditions familiar from extreme value theory, then (0.1) holds for some α–stable random variable S and properly chosen sequences (a_n) and (b_n). These conditions are quite restrictive however, even excluding some m-dependent sequences. For strongly mixing random sequences, a necessary and sufficient condition was obtained in Denker and Jakubowski [27] for the weak convergence of partial sums towards an α–stable distribution. Later, in [24] Davis and Hsing showed, by point process methods, that sequences which satisfy a regular variation condition for some $\alpha \in (0, 2)$ and certain mixing conditions also satisfy (0.1) with an α–stable limit. Building upon the same point process approach, Davis and Mikosch [25] generalized these results to multivariate sequences. Most recently, Bartkiewicz et al. [6] provided a detailed study of the conditions for the convergence of the partial sums of a strictly stationary process to an infinite variance stable distribution. They also determined the parameters of the limiting distribution in terms of some tail characteristics of the underlying stationary sequence.

The asymptotic behavior of the processes $V_n(\,\cdot\,)$ as $n \to \infty$ is an extensively studied subject in the probability literature. In our considerations the index of regular variation

α will be less than 2, which implies the variance of X_1 is infinite. In the finite-variance case, functional limit theorems differ considerably and have been investigated in greater depth, see for instance Billingsley [12], Herrndorf [37], Merlevède and Peligrad [49], and Peligrad and Utev [55].

A very readable proof of the functional limit theorem for i.i.d. sequences of regularly varying random variables with infinite variance can be found in Resnick [60]. Durrett and Resnick [28] considered functional limit theorems for dependent random variables in the context of martingale theory, while Leadbetter and Rootzén [45] studied this question in the context of extreme value theory. Their functional limit theorems hold in Skorohod's J_1 topology. However, this choice of topology excludes many important applied models. Avram and Taqqu [3] obtained a functional limit theorem in $D[0,1]$ endowed with Skorohod's M_1 topology for sums of MA processes with nonnegative coefficients. They also showed why the J_1 metric is not always well suited for studying weak convergence of the processes V_n when the variables X_n are not independent. For some more recent articles with related but somewhat different subjects we refer to Sly and Heyde [66] who obtained nonstandard limit theorems for functionals of regularly varying sequences with long-range Gaussian dependence structure sequences, and also to Aue et al. [1] who investigated the limit behavior of the functional CUSUM statistic and its randomly permuted version for i.i.d. random variables which are in the domain of attraction of a strictly α–stable law, for $\alpha \in (0,2)$.

The main result of this thesis shows that for a strictly stationary, regularly varying sequence for which clusters of high-threshold excesses can be broken down into asymptotically independent blocks, the properly centered partial sum process $(V_n(t))_{t\in[0,1]}$ converges to an α–stable Lévy process in the space $D[0,1]$ endowed with Skorohod's M_1 metric under the condition that all extremes within one such cluster have the same sign. In proving this result we combine some ideas used in the i.i.d. case by Resnick [58, 60] with a new point process convergence result and some particularities of the M_1 metric on $D[0,1]$ that can be deduced from Whitt [69]. This result can be viewed as a generalization of results in Leadbetter and Rootzén [45], where clustering of extremes is essentially prohibited, and in Avram and Taqqu [3].

The thesis is organized as follows. In Chapter 1 we introduce notions and tools which we are going to use in studying functional limit theorems. In Section 1.1 we define and list some basic properties of vague convergence of measures. Using this concept, in Section 1.2 we introduce regular variation and list some well known results connected with it. The property of regular variation, which will be the key notion in our considerations, has been studied extensively in the past; see for instance Bingham et al. [13], de Haan [35], de Haan and Resnick [36], Resnick [58] and Rvačeva [61]. Section 1.3 gives one characterization of regularly varying processes in terms of their tail processes; see Basrak and Segers [10]. In Section 1.4 is given a brief introduction to point process theory. The emphasis is on convergence in distribution of point processes and its characterization by convergence of corresponding Laplace functionals. A special attention is given to the Poisson random measure which will play an important role in the subsequent chapters. For a detailed overview on point processes we refer to Kallenberg [39]. Section 1.5 introduces several concepts of dependence, including α–mixing, ρ–mixing, and a new mixing condition, namely the mixing condition $\mathcal{A}'(a_n)$, which will be used in functional limit theorems as a measure of dependence. In Section 1.6 we present a result concerning the convergence in distribution of a special type of point processes, which will be used in the proofs of our main results. The limits in our functional limit theorems will be Lévy processes, and therefore Section 1.7 contains some basic notions and results from the theory of Lévy processes.

Chapter 2 is the main part of this thesis. In Section 2.1 we introduce the space $D[0,1]$, which will serve as the space of sample paths of stochastic processes we will consider. We equip this space with Skorohod's M_1 and J_1 metrics and discus the differences between them. We refer to Whitt [68, 69] for a detailed discussion on this concepts. Section 2.2 is concerned with the summation functional and its continuity with respect to the M_1 topology. In Section 2.3 is proven the main result of this thesis. It gives conditions under which the partial sum stochastic process $V_n(\,\cdot\,)$ converges in distribution to a stable Lévy process in $D[0,1]$ under the M_1 topology. It also characterizes the limiting process in terms of its characteristic triple. Section 2.4 provides discussion about the conditions and conclusions of the theorem proven in the previous section. It gives also a explanation why the M_1 topology can not be replaced here by the J_1 topology.

In Chapter 3 we deal with cases when functional limit theorems hold with the J_1 topology. In Section 3.1 we present the result by Resnick [60] which gives the functional limit theorem for independent and identically distributed regularly varying random variables. This result is partially used in Section 3.2 in deriving the corresponding functional limit theorem for the case when the random variables are dependent, but have isolated extremes. Another case when the J_1 topology is good enough is when we alter the definition of the partial sum process in an appropriate way. This is the content of Section 3.3.

Chapter 4 is devoted to some particular time series models, namely MA, GARCH, ARMA and stochastic volatility models. To these models we apply the obtained results and obtain sufficient conditions for functional limit theorems to hold for each of these models.

Chapter 1

Notions and tools

In this chapter we introduce notions and tools that serve as a base for the subsequent chapters. In particular, we put our attention on regular variation, point processes and convergence of point processes under weak dependence. We also collect various results concerning these notions. Some of them are of independent interest, but all of them will play an important role in proving functional limit theorems in the following chapters.

1.1 Vague convergence

Let $\mathbb{E} = \overline{\mathbb{R}} \setminus \{0\}$, where $\overline{\mathbb{R}} = [-\infty, \infty]$. For $x, y \in \mathbb{E}$ define

$$\rho(x,y) = \max\left\{ \left| \frac{1}{|x|} - \frac{1}{|y|} \right|, |\operatorname{sign} x - \operatorname{sign} y| \right\}. \tag{1.1}$$

Then $(\mathbb{E}, \rho)$ is a locally compact, complete and separable metric space, and

$$\mathcal{B}(\mathbb{R}) \cap (\mathbb{R} \setminus \{0\}) = \mathcal{B}(\mathbb{E}) \cap (\mathbb{R} \setminus \{0\}),^{1} \tag{1.2}$$

where $\mathcal{B}(\mathbb{E})$ denotes the Borel σ-algebra generated by the ρ-open sets, while $\mathcal{B}(\mathbb{R})$ denotes the standard Borel σ-algebra generated by the open sets in the euclidian topology (for a proof of these statements we refer to Theorem 1.5 in Lindskog [48]). Relation

[1]For $B \subseteq X$ we put $\mathcal{B}(X) \cap B := \{A \cap B : A \in \mathcal{B}(X)\}$.

(1.2) tells us that on $\mathbb{R} \setminus \{0\}$ the Borel σ-algebra $\mathcal{B}(\mathbb{E})$ coincides with the usual Borel σ-algebra $\mathcal{B}(\mathbb{R})$, i.e. the Borel sets we are interested in are the usual Borel sets (the points of $\overline{\mathbb{R}} \setminus \mathbb{R}$ will be of no interest apart from being a part of the modified state space which will enable us to use the notion of vague convergence). We say a set B is bounded away from origin if $0 \notin \overline{B}$, where $\overline{B}$ denotes the closure of B.

Proposition 1.1. *Every set $B \in \mathcal{B}(\mathbb{R})$ bounded away from origin (in the euclidian topology) is relatively compact, i.e. its closure $\overline{B}$ is compact (in the topology induced by the metric ρ).*

Proof. Assume $B \in \mathcal{B}(\mathbb{R})$ is bounded away from origin. Then $B \subseteq \mathbb{R} \setminus [-a, a]$ for some $a > 0$. Since in a complete metric space a subset is relative compact if and only if it s totally bounded (see for instance Theorem 0.25 in Folland [33]), it suffices to show that B is totally bounded, i.e. for every $\epsilon > 0$, B can be covered by finitely many balls of radius ϵ.[2] Write $B_1 = B \cap (a, \infty)$ and $B_2 = B \cap (-\infty, -a)$. Then $B = B_1 \cup B_2$. To show that B is totally bounded it is clear that it is sufficient to show that B_1 and B_2 are totally bounded. Let show this for B_1 (for B_2 it can be shown in the same way). We distinguish two cases:

(i) If $1/2\epsilon < a$, then
$$(a, \infty) \subseteq \Big(\frac{1}{2\epsilon}, \infty\Big) \subseteq K_\rho\Big(\frac{1}{\epsilon}, \epsilon\Big).$$

(ii) If $1/2\epsilon \geqslant a$, let $x_1 < a \leqslant x_2 < x_3 < \ldots < x_{m-1} < x_m = 1/2\epsilon$ such that $x_i - x_{i-1} = a^2\epsilon/3$ for every $i = 2, \ldots, m$. Then
$$[x_{i-1}, x_i] \subseteq \Big(\frac{x_i}{1 + x_i\epsilon}, x_i\Big] \subseteq K_\rho(x_i, \epsilon),$$
for every $i = 2, \ldots, m$, which implies
$$(a, \infty) \subseteq K_\rho\Big(\frac{1}{\epsilon}, \epsilon\Big) \cup K_\rho(x_2, \epsilon) \cup \cdots \cup K_\rho(x_m, \epsilon).$$

[2] If $x \in \mathbb{E}$ and $r > 0$, the (open) ball of radius r about x is the set $K_\rho(x, r) = \{y \in \mathbb{E} : \rho(x, y) < r\}$.

In both cases B_1 is covered by finitely many balls of radius ϵ. Therefore B_1 is totally bounded. This completes the proof. $\square$

A nonnegative measure μ on $(\mathbb{E}, \mathcal{B}(\mathbb{E}))$ is called *Radon* if $\mu(B) < \infty$ for all relatively compact $B \in \mathcal{B}(\mathbb{E})$. Define

$$M_+(\mathbb{E}) = \{\mu : \mu \text{ is a Radon measure on } (\mathbb{E}, \mathcal{B}(\mathbb{E}))\}.$$

On the set $M_+(\mathbb{E})$ we introduce a topology in the following way. Let $C_K^+(\mathbb{E})$ denote the class of all nonnegative continuous real functions on $\mathbb{E}$ with compact support,[3] i.e.

$$C_K^+(\mathbb{E}) = \{f\colon \mathbb{E} \to [0, \infty) : f \text{ is continuous with compact support}\}.$$

The class of all finite intersections of sets of the form $\{\mu \in M_+(\mathbb{E}) : a < \int_{\mathbb{E}} f(x)\,\mu(dx) <$

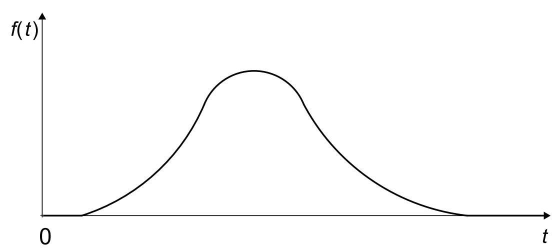

Figure 1.1: *An example of a function from* $C_K^+(\mathbb{E})$.

$b\}$ with arbitrary $f \in C_K^+(\mathbb{E})$ and $a, b \in \mathbb{R}$ form a base for a topology on $M_+(\mathbb{E})$. The topology with this base is called the *vague topology*. This topology is metrizable and one measure that induces this topology is given by

$$d_v(\mu_1, \mu_2) = \sum_{k=1}^{\infty} 2^{-k} \Big[1 - \exp\Big\{ - \Big| \int_{\mathbb{E}} f_k(x)\,\mu_1(dx) - \int_{\mathbb{E}} f_k\,\mu_2(dx) \Big| \Big\} \Big], \tag{1.3}$$

for some sequence of functions $f_k \in C_K^+(\mathbb{E})$. This metrization is complete and separable (see Kallenberg [39, p. 170]). Note that a sequence (μ_n) of measures in $M_+(\mathbb{E})$ converges

[3]The support of a function f is the set $\mathrm{supp}(f) = \overline{\{x \in \mathbb{E} : f(x) \neq 0\}}$.

to $\mu \in M_+(\mathbb{E})$ in the vague topology (or *vaguely*: written $\mu_n \xrightarrow{v} \mu$) if and only if $\int_{\mathbb{E}} f(x)\, \mu_n(dx) \to \int_{\mathbb{E}} f(x)\, \mu(dx)$ for every $f \in C_K^+(\mathbb{E})$.

This convergence may hold also for some non-continuous functions, which is the statement of the following result (see Kallenberg [39], 15.7.3).

Proposition 1.2. *Let $\mu, \mu_1, \mu_2, \ldots \in M_+(\mathbb{E})$ with $\mu_n \xrightarrow{v} \mu$ as $n \to \infty$. Then, as $n \to \infty$, $\int_{\mathbb{E}} f(x)\, \mu_n(dx) \to \int_{\mathbb{E}} f(x)\, \mu(dx)$, for every bounded measurable function $f\colon \mathbb{E} \to [0, \infty)$ with compact support satisfying $\mu(D_f) = 0$.*[4]

If we replace the space $M_+(\mathbb{E})$ by the subspace consisting of all finite measures in $M_+(\mathbb{E})$, and $C_K^+(\mathbb{E})$ by the class of all bounded continuous real functions on $\mathbb{E}$, we obtain the *weak topology*. Then a sequence (μ_n) converges to μ in this topology (or *weakly*: written $\mu_n \xrightarrow{w} \mu$) if and only if $\int_{\mathbb{E}} f(x)\, \mu_n(dx) \to \int_{\mathbb{E}} f(x)\, \mu(dx)$ for every bounded and continuous function $f\colon \mathbb{E} \to [0, \infty)$. It is obvious that weak convergence implies vague convergence. By the following result, for finite measures under an additional condition, the converse also holds (see Kallenberg [39], 15.7.6).

Proposition 1.3. *Suppose $\mu, \mu_1, \mu_2, \ldots \in M_+(\mathbb{E})$ are bounded measures. Then $\mu_n \xrightarrow{w} \mu$ if and only if $\mu_n \xrightarrow{\nu} \mu$ and $\mu_n(\mathbb{E}) \to \mu(\mathbb{E})$.*

Vague convergence is equivalent with convergence of measures of a special class of relatively compact sets. The precise statement is given in the following theorem (for a proof see Kallenberg [39], 15.7.2).

Theorem 1.4. *Let $\mu, \mu_1, \mu_2, \ldots \in M_+(\mathbb{E})$. Then the following statements are equivalent.*

(i) $\mu_n \xrightarrow{v} \mu$,

[4] D_f is the set of discontinuity points of the function f.

(ii) $\mu_n(B) \to \mu(B)$ *for every relatively compact set* $B \in \mathcal{B}(\mathbb{E})$ *such that* $\mu(\partial B) = 0$*,*

(iii) $\limsup_{n\to\infty} \mu_n(F) \leqslant \mu(F)$ *and* $\liminf_{n\to\infty} \mu_n(G) \geqslant \mu(G)$ *for every compact* $F \in \mathcal{B}(\mathbb{E})$ *and every open relatively compact* $G \in \mathcal{B}(\mathbb{E})$*.*

All notions introduced in this section can be generalized to the multidimensional case. Our state space, in d-dimensional case, then become $\mathbb{E}^d = \overline{\mathbb{R}}^d \setminus \{\mathbf{0}\}$. Then there exists a metric $\overline{\rho}$ on $\mathbb{E}^d$ such that $(\mathbb{E}^d, \overline{\rho})$ is a locally compact, complete and separable metric space. In an obvious manner we can directly generalize all the remaining notions and results in this section to the d-dimensional case (for details see Kallenberg [39] or Lindskog [48]). The concept of vague convergence can be generalized to an arbitrary locally compact topological space with countable base (see Kallenberg [39] or Resnick [60]).

1.2 Regular variation

In this section we will introduce the notion of regular variation for random vectors and give some basic results concerning regular variation. Next we will generalize this notion to sequences of random variables.

Definition 1.5. *A* d*-dimensional random vector* $\mathbf{X}$ *is* ***regularly varying*** *if there exists a sequence* (a_n) *of positive*[5] *real numbers tending to* ∞ *and a nonzero Radon measure* μ *on* $(\mathbb{E}^d, \mathcal{B}(\mathbb{E}^d))$ *with* $\mu(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ *such that, as* $n \to \infty$*,*

$$n\,\mathrm{P}\Big(\frac{\mathbf{X}}{a_n} \in \cdot\Big) \xrightarrow{v} \mu(\cdot). \tag{1.4}$$

Many facts about regular variation are known from the works of Feller [32], de Haan [35], de Haan and Resnick [36], Resnick [58] and Rvačeva [61]. More recent references

[5] A real number x is positive if $x > 0$, and negative if $x < 0$.

are for instance Basrak [7], Basrak et al. [8] and Lindskog [48]. Here we give a theorem in which we collect some of these facts that we are going to use in the following sections. First we give an equivalent formulation of regular variation in terms of weak convergence of finite measures on $(\mathbb{S}^{d-1}, \mathcal{B}(\mathbb{S}^{d-1}))$, where $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^d$, with $\|\cdot\|$ being an arbitrary (and fixed) norm on $\mathbb{R}^d$. Second we write down some basic properties of the limiting measure μ from (1.4).

Theorem 1.6. *(i) Let $\mathbf{X}$ be an $\mathbb{R}^d$-valued random vector. Then $\mathbf{X}$ is regularly varying if and only if there exist an $\alpha > 0$ and a probability measure σ on $\mathcal{B}(\mathbb{S}^{d-1})$ such that, for every $x > 0$, as $u \to \infty$,*

$$\frac{\mathrm{P}(\|\mathbf{X}\| > ux, \mathbf{X}/\|\mathbf{X}\| \in \cdot)}{\mathrm{P}(\|\mathbf{X}\| > u)} \xrightarrow{w} x^{-\alpha}\sigma(\cdot), \tag{1.5}$$

(ii) The limiting measure μ in (1.4) has the following property: there exists an $\alpha > 0$ such that

$$\mu(uB) = u^{-\alpha}\mu(B)$$

for every $u > 0$ and $B \in \mathcal{B}(\mathbb{E}^d)$.[6]

(iii) For the measure μ it also holds that

(1) $\mu(u\mathbb{S}^{d-1}) = 0$ for every $u > 0$;

(2) $\mu(\{\mathbf{x}\}) = 0$ for every $\mathbf{x} \in \mathbb{E}^d$;

(3) $\mu(\partial V_{u,S}) = \mu(V_{u,\partial S})$ for every $u > 0$ and $S \in \mathcal{B}(\mathbb{S}^{d-1})$, where $V_{u,S} = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| > u,\ \mathbf{x}/\|\mathbf{x}\| \in S\}$;

(4) $\mu(V_{0,\{\mathbf{s}\}}) = 0$ for all but at most countably many $\mathbf{s} \in \mathbb{S}^{d-1}$.

For the proof of this theorem we refer to Theorem 2.1.8 in Basrak [7] and theorems 1.8, 1.14 and 1.15 in Lindskog [48].

[6] This α is the same as in (1.5).

Definition 1.7. *The number α in (1.5) is called the* ***index of regular variation*** *of* $\mathbf{X}$*, while the probability measure σ is called the* ***spectral measure*** *of* $\mathbf{X}$.

Proposition 1.8. *Suppose the random vector $\mathbf{X}$ is regularly varying with index of regular variation $\alpha > 0$. Then*

$$\mathrm{P}(\|\mathbf{X}\| > x) = x^{-\alpha} L(x) \quad \text{for every } x > 0, \tag{1.6}$$

where L is a slowly varying function, i.e. for every $t > 0$, $L(tx)/L(x) \to 1$ as $x \to \infty$.

Proof. We have to prove that the function L defined by $L(x) = x^{\alpha}\mathrm{P}(\|\mathbf{X}\| > x)$ is slowly varying. Take an arbitrary $t > 0$. Then using relation (1.5) we obtain

$$\frac{L(tx)}{L(x)} = t^{\alpha} \cdot \frac{\mathrm{P}(\|\mathbf{X}\| > tx)}{\mathrm{P}(\|\mathbf{X}\| > x)} \to t^{\alpha} \cdot t^{-\alpha} = 1,$$

as $x \to \infty$. ☐

Remark 1.9. The sequence (a_n) that appears in Definition 1.5 is not unique. However, it satisfies the following asymptotic relation

$$\frac{a_{\lfloor \lambda n \rfloor}}{a_n} \to \lambda^{1/\alpha} \quad \text{as } n \to \infty.$$

Therefore it can be represented as

$$a_n = n^{1/\alpha} L'(n),$$

where L' is a slowly varying function. We shall often choose the sequence (a_n) such that $n\mathrm{P}(\|\mathbf{X}\| > a_n) \to 1$ as $n \to \infty$ (it suffices to take a_n to be the $1 - 1/n$ quantile of the distribution function of $\|\mathbf{X}\|$, for $n \geqslant 2$).

Remark 1.10. Rewriting the statement of Theorem 1.6 (i) in the 1-dimensional case we obtain that the random variable X is regularly varying if and only if there exist $\alpha > 0$ and $p \in [0,1]$ such that, for every $x > 0$, as $u \to \infty$,

$$\frac{\mathrm{P}(X > ux)}{\mathrm{P}(|X| > u)} \to px^{-\alpha} \quad \text{and} \quad \frac{\mathrm{P}(X < -ux)}{\mathrm{P}(|X| > u)} \to qx^{-\alpha}, \tag{1.7}$$

where $q = 1 - p$. The limit measure μ from relation (1.4) is then of the form

$$\mu(dx) = (p\alpha x^{-\alpha-1} 1_{(0,\infty)}(x) + q\alpha(-x)^{-\alpha-1} 1_{(-\infty,0)}(x))\, dx, \tag{1.8}$$

while the spectral measure σ is given by $\sigma(\{1\}) = p$ and $\sigma(\{-1\}) = q$.

If $X_1, \ldots, X_n$ are independent and identically distributed regularly varying random variables with index of regular variation $\alpha > 0$ then it is known that the n-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is regularly varying with the same index and his spectral measure concentrates on the points of intersection of the unit sphere with the axes. For the sake of illustration we shall prove this here for $n = 2$ (see Lemma 7.2 in Resnick [60] for a different proof of this result, but for nonnegative components only). For $r > 0$ let $B_r = \{x \in \mathbb{R} : |x| < r\}$.

Proposition 1.11. *Suppose X_1 and X_2 are independent and identically distributed random variables, regularly varying with index $\alpha > 0$. Then the random vector $\mathbf{X} = (X_1, X_2)$ is regularly varying with index α, and the spectral measure of $\mathbf{X}$ is concentrated on the following points* $(1,0), (0,1), (-1,0),\ (0,-1)$.

Proof. Take the sequence (a_n) such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. Then from the setting in Remark 1.10 it follows that, for any $r > 0$,

$$n\mathrm{P}(|X_1| > ra_n) \to r^{-\alpha} \quad \text{as } n \to \infty.$$

Take now $\epsilon_1 > 0$ and $\epsilon_2 > 0$. Since

$$\begin{aligned} n\mathrm{P}(a_n^{-1}\mathbf{X} \in B_{\epsilon_1}^c \times B_{\epsilon_2}^c) &= n\mathrm{P}(|X_1| \geqslant \epsilon_1 a_n)\mathrm{P}(|X_2| \geqslant \epsilon_2 a_n) \\ &\to 0, \end{aligned}$$

as $n \to \infty$, it follows that $n\mathrm{P}(a_n^{-1}\mathbf{X} \in \cdot) \xrightarrow{v} \widehat{\mu}(\cdot)$ as $n \to \infty$, where $\widehat{\mu}$ is a Radon measure on $(\mathbb{E}^2, \mathcal{B}(\mathbb{E}^2))$ which concentrates on $(\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$ and for any $B \in \mathcal{B}(\mathbb{E})$, $\widehat{\mu}(\{0\} \times B) = \widehat{\mu}(B \times \{0\}) = \mu(B)$, where μ is the limiting measure of regular variation of X_1 (and X_2), i.e. $n\mathrm{P}(a_n^{-1}X_1 \in \cdot) \xrightarrow{v} \mu(\cdot)$. For an arbitrary $r > 0$, by Theorem 1.6 (ii), we have

$$\widehat{\mu}((r,\infty) \times \mathbb{E}) = \widehat{\mu}((r,\infty) \times \{0\}) = \mu((r,\infty)) = r^{-\alpha}\mu((1,\infty)).$$

In particular, for $r = 1$, $\widehat{\mu}((1,\infty) \times \mathbb{E}) = \mu((1,\infty))$. Therefore

$$\widehat{\mu}((r,\infty) \times \mathbb{E}) = r^{-\alpha}\widehat{\mu}((1,\infty) \times \mathbb{E}),$$

and we may conclude that the index of regular variation of $\mathbf{X}$ is α.

Assume now the spectral measure σ of X_1 (and X_2) is of the form $\sigma(\{1\}) = p \in [0,1]$ and $\sigma(\{-1\}) = q = 1 - p$. Let us find the spectral measure $\widehat{\sigma}$ of $\mathbf{X}$. Let $\|\cdot\|$ be the so-called "sup" norm on $\mathbb{R}^2$, i.e. $\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}$. First of all note that from $n\mathrm{P}(|X_i| > a_n) \to 1$ as $n \to \infty$ for $i = 1, 2$, it follows that $n\mathrm{P}(\|\mathbf{X}\| > a_n) \to 2$. Thus by relation (1.5) we obtain that, as $n \to \infty$,

$$n\mathrm{P}(\|\mathbf{X}\| > a_n,\, \mathbf{X}/\|\mathbf{X}\| \in S) \to 2\widehat{\sigma}(S), \tag{1.9}$$

for every $S \in \mathcal{B}(\mathbb{S}^1)$ such that $\widehat{\sigma}(\partial S) = 0$. Since for all but at most countably many $\mathbf{s} \in \mathbb{S}^1$, $\widehat{\sigma}(\{\mathbf{s}\}) = 0$, we can find a sequence of points (x_k, y_k) in $\mathbb{S}^1 \cap \{(x,y) \in \mathbb{R}^2 : x, y > 0\}$ such that $\widehat{\sigma}(\{(x_k, y_k)\}) = \widehat{\sigma}(\{(x_k, -y_k)\}) = 0$ for every $k \in \mathbb{N}$ and $(x_k, y_k) \to (1, 0)$ as $k \to \infty$ in the $\|\cdot\|$ norm. Let S_k be the connected closed subset of $\mathbb{S}^1$ with edges in the points (x_k, y_k) and $(x_k, -y_k)$ which contains the point $(1,0)$, i.e. $S_k = \{(x,y) \in \mathbb{S}^1 : x \geqslant x_k\}$. Then, since $\widehat{\sigma}(\partial S_k) = \widehat{\sigma}(\{(x_k, y_k), (x_k, -y_k)\}) = 0$, from relation (1.9) we obtain that, as $n \to \infty$,

$$n\mathrm{P}(\|\mathbf{X}\| > a_n,\, \mathbf{X}/\|\mathbf{X}\| \in S_k) \to 2\widehat{\sigma}(S_k), \qquad \text{for all } k \in \mathbb{N}.$$

On the other hand, since $\widehat{\mu}$ is concentrated on the axes, by Theorem 1.6 (iii) we have $\widehat{\mu}(\partial V_{1,S_k}) = \widehat{\mu}(V_{1,\partial S_k}) = 0$. Hence

$$\begin{aligned} n\mathrm{P}(\|\mathbf{X}\| > a_n,\, \mathbf{X}/\|\mathbf{X}\| \in S_k) &= n\mathrm{P}(a_n^{-1}\mathbf{X} \in V_{1,S}) \\ &\to \widehat{\mu}(V_{1,S}) = \widehat{\mu}((1,\infty) \times \{0\}), \end{aligned}$$

as $n \to \infty$. Comparing the last two equations we conclude that

$$\widehat{\mu}((1,\infty) \times \{0\}) = 2\widehat{\sigma}(S_k)$$

for all $k \in \mathbb{N}$. Since

$$\widehat{\mu}((1,\infty) \times \{0\})) = \mu((1,\infty)) = \lim_{n\to\infty} n\mathrm{P}(X_1 > a_n) = p,$$

it follows that $\widehat{\sigma}(S_k) = p/2$ for all $k \in \mathbb{N}$. Since the sets S_k, as k tends to ∞, form a decreasing sequence that shrinks to the point $(1,0)$, using the continuity probability property with respect to a decreasing sequence of sets, we obtain that $\widehat{\sigma}(\{(1,0)\}) = p/2$, and in the same manner, $\widehat{\sigma}(\{(0,1)\}) = p/2$, $\widehat{\sigma}(\{(-1,0)\}) = q/2$ and $\widehat{\sigma}(\{(0,-1)\}) = q/2$. In particular we proved the spectral measure is concentrated on the points of intersection of the unit sphere $\mathbb{S}^1$ with the axes. $\square$

In regular variation theory an important role plays the relation between the tails and the truncated moments of regularly varying random variables. The following result gives this relation which we shall use a couple of times in forthcoming chapters. It is based on Karamata's theorem (Theorem 1 in Feller [32, p. 273]; see also Theorem 1.5.11 in Bingham et al. [13]).

Theorem 1.12. *Suppose the random variable X is regularly varying with index of regular variation $\alpha > 0$. Then, as $x \to \infty$,*

$$\frac{\mathrm{E}\big(|X|^\lambda 1_{\{|X|>x\}}\big)}{x^\lambda \mathrm{P}(|X|>x)} \to \frac{\alpha}{\alpha-\lambda} \qquad \text{if } \ 0<\lambda<\alpha, \tag{1.10}$$

and

$$\frac{\mathrm{E}\big(|X|^\lambda 1_{\{|X|\leqslant x\}}\big)}{x^\lambda \mathrm{P}(|X|>x)} \to \frac{\alpha}{\lambda-\alpha} \qquad \text{if } \ \lambda>\alpha. \tag{1.11}$$

Proof. If $\lambda \in (0,\alpha)$, then by Lemma 5.7 in Durrett [29, p. 43] and Theorem 1 in Feller [32] we have

$$\begin{aligned}
\frac{\mathrm{E}\big(|X|^\lambda 1_{\{|X|>x\}}\big)}{x^\lambda \mathrm{P}(|X|>x)} &= \frac{\int_0^\infty \lambda y^{\lambda-1}\mathrm{P}\big(|X|1_{\{|X|>x\}}>y\big)\,dy}{x^\lambda \mathrm{P}(|X|>x)} \\
&= \frac{\int_0^x \lambda y^{\lambda-1}\mathrm{P}\big(|X|>x\big)\,dy}{x^\lambda \mathrm{P}(|X|>x)} + \frac{\int_x^\infty \lambda y^{\lambda-1}\mathrm{P}\big(|X|>y\big)\,dy}{x^\lambda \mathrm{P}(|X|>x)} \\
&= 1+\lambda\frac{\int_x^\infty y^{\lambda-1}\mathrm{P}\big(|X|>y\big)\,dy}{x^\lambda \mathrm{P}(|X|>x)} \\
&\to 1+\frac{\lambda}{\alpha-\lambda} = \frac{\alpha}{\alpha-\lambda} \qquad \text{as } x\to\infty.
\end{aligned}$$

If $\lambda > \alpha$ then using again Lemma 5.7 in [29] and the fact that for $y \in (0,x)$

$$\begin{aligned}
\mathrm{P}(|X|>y) &= \mathrm{P}(|X|>y,\,|X|>x) + \mathrm{P}(|X|>y,\,|X|\leqslant x) \\
&= \mathrm{P}(|X|>x) + \mathrm{P}\big(|X|1_{\{|X|\leqslant x\}}>y\big),
\end{aligned}$$

we obtain that

$$\begin{aligned}
\frac{\mathrm{E}\big(|X|^\lambda 1_{\{|X|\leqslant x\}}\big)}{x^\lambda \mathrm{P}(|X|>x)} &= \frac{\int_0^\infty \lambda y^{\lambda-1}\mathrm{P}\big(|X|1_{\{|X|\leqslant x\}}>y\big)\,dy}{x^\lambda \mathrm{P}(|X|>x)} \\
&= \frac{\int_0^x \lambda y^{\lambda-1}\mathrm{P}\big(|X|1_{\{|X|\leqslant x\}}>y\big)\,dy}{x^\lambda \mathrm{P}(|X|>x)} \\
&= \frac{\int_0^x \lambda y^{\lambda-1}\mathrm{P}\big(|X|>y\big)\,dy}{x^\lambda \mathrm{P}(|X|>x)} - \frac{\int_0^x \lambda y^{\lambda-1}\mathrm{P}\big(|X|>x\big)\,dy}{x^\lambda \mathrm{P}(|X|>x)} \\
&= \lambda\frac{\int_0^x y^{\lambda-1}\mathrm{P}\big(|X|>y\big)\,dy}{x^\lambda \mathrm{P}(|X|>x)} - 1.
\end{aligned}$$

Now an application of the above mentioned Theorem 1 in [32] yields that

$$\frac{\mathrm{E}\big(|X|^{\lambda}1_{\{|X|\leqslant x\}}\big)}{x^{\lambda}\mathrm{P}(|X|>x)} \to \frac{\lambda}{\lambda-\alpha}-1=\frac{\alpha}{\lambda-\alpha} \qquad \text{as } x\to\infty.$$

□

Next we define the regular variation property for random processes.

Definition 1.13. *We say a random process $(X_n)_{n\in\mathbb{Z}}$ is **regularly varying** with index $\alpha>0$ if all its finite-dimensional distributions are regularly varying with index α.*

Remark 1.14. From Definition 1.13 we have immediately that a strictly stationary sequence of random variables (X_n) is regularly varying with index $\alpha>0$ if for every $k\in\mathbb{N}$ the random vector $(X_1,\dots,X_k)$ is regularly varying with index α. In particular, for every n the random variable X_n is regularly varying with index α.

Remark 1.15. From Proposition 1.11 (and its generalization to arbitrary finite-dimensional random vectors with independent, identically distributed and regularly varying components) it follows that if (X_n) is a sequence of i.i.d. regularly varying random variables with index $\alpha>0$, then the random process (X_n) is regularly varying with index α.

1.3 Tail process

The following result provides a useful characterization of regular variation for strictly stationary processes.

Theorem 1.16. *(Basrak and Segers [10], Theorem 2.1) Let $(X_n)_{n\in\mathbb{Z}}$ be a strictly stationary process in $\mathbb{R}$ and let $\alpha\in(0,\infty)$. Then (X_n) is regularly varying of index α if and only if there exists a process $(Y_n)_{n\in\mathbb{Z}}$ in $\mathbb{R}$ with $\mathrm{P}(|Y_0|>y)=y^{-\alpha}$ for $y\geqslant 1$ such that, as $x\to\infty$,*

$$\big((x^{-1}X_n)_{n\in\mathbb{Z}}\,\big|\,|X_0|>x\big) \xrightarrow{\text{fidi}} (Y_n)_{n\in\mathbb{Z}}, \tag{1.12}$$

where "$\xrightarrow{\text{fidi}}$" denotes convergence of finite-dimensional distributions.

Definition 1.17. *The process $(Y_n)_{n\in\mathbb{Z}}$ that appears in Theorem 1.16 is called the **tail process** of a strictly stationary regularly varying random process $(X_n)_{n\in\mathbb{Z}}$.*

Write $\Theta_n = Y_n/|Y_0|$ for $n \in \mathbb{Z}$. If $(X_n)_{n\in\mathbb{Z}}$ is regularly varying, by Corollary 3.2 in Basrak and Segers [10]

$$\big((|X_0|^{-1}X_n)_{n\in\mathbb{Z}} \,\big|\, |X_0| > x\big) \xrightarrow{\text{fidi}} (\Theta_n)_{n\in\mathbb{Z}}. \tag{1.13}$$

The process $(\Theta_n)_{n\in\mathbb{Z}}$ is independent of $|Y_0|$ (see Theorem 3.1 in [10]), and by relations (1.5) and (1.13), the law of $\Theta_0 \in \mathbb{S}^0 = \{-1, 1\}$ is the spectral measure of X_0. Thus we call the process $(\Theta_n)_{n\in\mathbb{Z}}$ the *spectral process* of $(X_n)_{n\in\mathbb{Z}}$.

Example 1.18. Suppose (X_n) is a strictly stationary regularly varying process with index $\alpha > 0$ consisting of independent random variables. Its tail process has a very simple representation. Fix $k \in \mathbb{Z} \setminus \{0\}$ and let $r > 0$ arbitrary. From relation (1.12) we obtain that

$$\begin{aligned} \mathrm{P}(|Y_k| > r) &= \lim_{n\to\infty} \mathrm{P}(|X_k| > ra_n \,\big|\, |X_0| > a_n) = \lim_{n\to\infty} \mathrm{P}(|X_k| > ra_n) \\ &= 0, \end{aligned}$$

where the last equation is a direct consequence of relation (1.4). Since $r > 0$ is arbitrary it follows that $Y_k = 0$ for every $k \in \mathbb{Z} \setminus \{0\}$. From relation (1.12) and the regular variation property of X_0 we obtain that $\mathrm{P}(|Y_0| > y) = y^{-\alpha}$ for $y \geqslant 1$. Therefore $Y_0 = \Theta_0|Y_0|$, where the law of Θ_0 is the spectral measure of X_0, $\ln|Y_0|$ has exponential distribution with parameter α, and Θ_0 and $|Y_0|$ are independent.

1.4 Point processes

In studding functional limit theory the notion of point process will be very useful. Our functional limit theorems will relay on convergence of a specific sequence of point processes. In this section we introduce the basic notions and results on point processes which will be used later on. For more background on the theory of point processes we refer to Kallenberg [39] and Resnick [60].

Let $\mathbb{X}$ be a locally compact Hausdorff topological space with countable base and $\mathcal{B}(\mathbb{X})$ its Borel σ-field. As in Section 1.1 denote by $M_+(\mathbb{X})$ the space of Radon measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ endowed with vague topology. A *Radon point measure* is an element of $M_+(\mathbb{X})$ of the form $m = \sum_{i=1}^{\infty} \delta_{x_i}$, where δ_x is the Dirac measure:

$$\delta_x(A) = \begin{cases} 1, & x \in A, \\ 0, & \text{otherwise}, \end{cases}$$

for every $A \in \mathcal{B}(\mathbb{X})$. Let $M_p(\mathbb{X})$ denote the set of all Radon point measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. Since $M_p(\mathbb{X})$ is a subset of $M_+(\mathbb{X})$ we endow it with the relative topology. Let $\mathcal{M}_p(\mathbb{X})$ be the Borel σ-field of subsets of $M_p(\mathbb{X})$ generated by open sets.

Definition 1.19. *A* ***point process*** *on $\mathbb{X}$ is a measurable mapping from a given probability space to the measurable space $(M_p(\mathbb{X}), \mathcal{M}_p(\mathbb{X}))$.*

Example 1.20. A standard example of point process is the Poisson process. Suppose μ is a given Radon measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$. We say N is a *Poisson process with mean (intensity) measure* μ or, synonymously, a *Poisson random measure* (PRM(μ)), if it satisfies the following conditions:

(i) for every $A \in \mathcal{B}(\mathbb{X})$ and nonnegative integer k,

$$\mathrm{P}(N(A) = k) = \begin{cases} \frac{e^{-\mu(A)}\mu(A)^k}{k!}, & \mu(A) < \infty, \\ 0, & \text{otherwise}, \end{cases}$$

(ii) if $A_1, \ldots, A_k$ are mutually disjoint subsets of $\mathbb{X}$, then $N(A_1), \ldots, N(A_k)$ are independent random variables.

When $\mathbb{X}$ is a finite-dimensional Euclidian space $\mathbb{R}^d$ or one of its topological subsets and the mean measure μ is a multiple of Lebesgue measure, we call the process *homogeneous*. Therefore in the homogeneous case, for any Borel set A, $N(A)$ is a Poisson random variable with mean $\mathrm{E}N(A) = \lambda\mathbb{LEB}(A)$, for some $\lambda > 0$, where $\mathbb{LEB}(A)$ is the Lebesgue measure of A. The parameter λ is called the *rate* (or the *intensity*) of N.

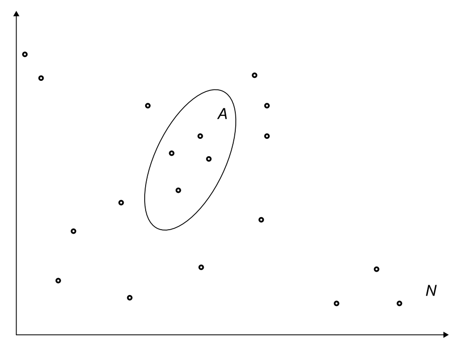

Figure 1.2: *Point processes can be viewed as collections of randomly placed points in the state space. Here is given an example of a point process N, yielding $N(A) = 4$.*

Next we define the notion of convergence in distribution for point processes in the usual way.

Definition 1.21. *We say a sequence of point processes (N_n) on $\mathbb{X}$* ***converges in distribution*** *to a point process N on $\mathbb{X}$, and write $N_n \xrightarrow{d} N$, if $\mathrm{E}f(N_n) \to \mathrm{E}f(N)$ for every bounded continuous function $f\colon M_p(\mathbb{X}) \to \mathbb{R}$.*

Let $\mathcal{B}_+$ denote the set of bounded measurable functions $f\colon \mathbb{X} \mapsto [0,\infty)$. For $f \in \mathcal{B}_+$ and $\mu \in M_p(\mathbb{X})$, we use the notation

$$\mu(f) = \int_{\mathbb{X}} f(x)\mu(dx).$$

Thus for $m = \sum_{i=1}^{\infty} \delta_{x_i} \in M_p(\mathbb{X})$,

$$m(f) = \int_{\mathbb{X}} f(x)m(dx) = \sum_{i=1}^{\infty} f(x_i).$$

In dealing with distributions of point processes a very useful transform technique is the Laplace functional.

Definition 1.22. *Let N be a point process on $\mathbb{X}$. The **Laplace functional** of N is the nonnegative function on $\mathcal{B}_+$ given by*

$$\Psi_N(f) = \mathrm{E}e^{-N(f)}, \quad f \in \mathcal{B}_+.$$

The distribution of a given point process N is uniquely determined by the values of its Laplace functional $\Psi_N(f)$, $f \in C_K^+(\mathbb{X})$ (see Theorem 3.1 in Kallenberg [39]). The next result gives the characterization of convergence in distribution of point processes by convergence of Laplace functionals on $C_K^+(\mathbb{X})$ (see Kallenberg [39], Theorem 4.2).

Theorem 1.23. *Let $N, N_1, N_2, \dots$ be point processes on $\mathbb{X}$. Then $N_n \xrightarrow{d} N$ if and only if $\Psi_{N_n}(f) \to \Psi_N(f)$ for every $f \in C_K^+(\mathbb{X})$.*

Example 1.24. Here we give the Laplace functionals in three special cases:

(1) Let $\mu_0 \in M_p(\mathbb{X})$. Define the point process N to be identically μ_0, i.e. if $(\Omega, \mathcal{F}, \mathrm{P})$ is the underlying probability space then $N(\omega) = \mu_0$ for every $\omega \in \Omega$. The Laplace functional of N at $f \in \mathcal{B}_+$ is then given by

$$\Psi_N(f) = \int_\Omega e^{-N(\omega)(f)}\, d\mathrm{P}(\omega) = e^{-\mu_0(f)}.$$

(2) Suppose $X_1, \dots, X_n$ are i.i.d. random elements in $\mathbb{X}$ and define the point process N by

$$N = \sum_{i=1}^{n} \delta_{X_i}.$$

Its Laplace functional is of the following form

$$\Psi_N(f) = \mathrm{E}e^{-N(f)} = \mathrm{E}e^{-\sum_{i=1}^n f(X_i)} = \left(\mathrm{E}e^{-f(X_1)}\right)^n, \quad f \in \mathcal{B}_+.$$

(3) The Laplace functional of a Poisson process N with mean measure μ can be calculated as

$$\Psi_N(f) = \exp\Big\{ - \int_{\mathbb{X}} (1 - e^{-f(x)})\,\mu(dx)\Big\}, \quad f \in \mathcal{B}_+$$

(see for instance Theorem 5.1 in Resnick [60]).

The next result tells us when we can enlarge the dimension of the points of the sequence of point processes which converges to a Poisson random measure and retain the Poisson structure in the limit. For a proof see Lemma 4.3 in Resnick [58].

Lemma 1.25. *Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be two locally compact Hausdorff topological spaces with countable base. Suppose (X_k) and $(Y_{n,k})$ are random elements of $\mathbb{X}_1$ and $\mathbb{X}_2$ respectively. If (X_k) is an i.i.d. sequence of random elements, such that for every $n \in \mathbb{N}$ the families (X_k) and $(Y_{n,k})_k$ are independent, and if, as $n \to \infty$,*

$$\sum_{k=1}^{n} \delta_{Y_{n,k}} \xrightarrow{d} \mathrm{PRM}(\mu),$$

on $\mathbb{X}_2$, then as $n \to \infty$,

$$\sum_{k=1}^{n} \delta_{(X_k, Y_{n,k})} \xrightarrow{d} \mathrm{PRM}(P(X_1 \in \cdot) \times \mu)$$

on $\mathbb{X}_1 \times \mathbb{X}_2$.

In Section 3.1 we will need the following two results. The first one gives a sufficient condition for two sequences of points processes to be close in probability with respect to the vague metric introduced in (1.3), while the second one describes the convergence of a point process formed from a triangular array of random variables to a Poisson random measure.

Proposition 1.26. *Suppose (N_n) and (N'_n) are two sequences of point processes on $\mathbb{X}$ such that for every $f \in C_K^+(\mathbb{X})$, as $n \to \infty$,*

$$|N_n(f) - N'_n(f)| \xrightarrow{P} 0.$$

Then $d_v(N_n, N'_n) \xrightarrow{P} 0$ as $n \to \infty$, where d_v is the vague metric given in (1.3).

Proof. Let ϵ, $\delta > 0$ be arbitrary. Since the series $\sum_{k=1}^{\infty} 2^{-k}$ converges, there exists an $k_0 = k_0(\delta) \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} 2^{-k} \leqslant \frac{\delta}{2}.$$

For every $k = 1, \ldots, k_0$ it holds that $|N_n(f_k) - N'_n(f_k)| \xrightarrow{P} 0$ as $n \to \infty$, where (f_k) are the functions from relation (1.3). Thus there exists an $n_0 = n_0(\epsilon, \delta) \in \mathbb{N}$ such that for every $k = 1, \ldots, k_0$,

$$\mathrm{P}\Big(|N_n(f_k) - N'_n(f_k)| > \frac{\delta}{2k_0}\Big) < \frac{\epsilon}{k_0}, \quad n \geqslant n_0.$$

Then using the inequality $1 - e^{-x} \leqslant x \wedge 1$ for $x \geqslant 0$, where $s \wedge t$ denotes $\min\{s, t\}$, we obtain for every $n \geqslant n_0$,

$$\begin{aligned}
\mathrm{P}(d_v(N_n, N'_n) > \delta) &= \mathrm{P}\Big(\sum_{k=1}^{\infty} 2^{-k}[1 - e^{-|N_n(f_k) - N'_n(f_k)|}] > \delta\Big) \\
&\leqslant \sum_{k=1}^{k_0} \mathrm{P}\Big(|N_n(f_k) - N'_n(f_k)| > \frac{\delta}{2k_0}\Big) + \mathrm{P}\Big(\sum_{k=k_0+1}^{\infty} 2^{-k} > \frac{\delta}{2}\Big) \\
&< k_0 \cdot \frac{\epsilon}{k_0} = \epsilon,
\end{aligned}$$

showing that $d_v(N_n, N'_n) \xrightarrow{P} 0$. $\square$

Proposition 1.27. *Suppose $(Y_{n,k})$ are random elements of $\mathbb{X}$, such that for every $n \in \mathbb{N}$, $Y_{n,1}, Y_{n,2}, Y_{n,3}, \ldots$ are i.i.d. and*

$$m_n \mathrm{P}(Y_{n,1} \in \cdot) \xrightarrow{v} \nu(\cdot), \tag{1.14}$$

as $n \to \infty$, where (m_n) is a sequence of nonnegative integers such that $m_n \to \infty$ and ν is a Radon measure. Then, as $n \to \infty$,

$$\sum_{k=1}^{m_n} \delta_{Y_{n,k}} \xrightarrow{d} \mathrm{PRM}(\nu)$$

on $\mathbb{X}$.

Since the proof of this proposition follows the same argument as given in the proof of Theorem 5.3 in Resnick [60], it is here omitted.

Let $\mathbb{X}'$ be a measurable subset of $\mathbb{X}$ and give $\mathbb{X}'$ the relative topology inherited from $\mathbb{X}$. Define the *restriction map* $T\colon M_p(\mathbb{X}) \to M_p(\mathbb{X}')$ by

$$Tm = m\big|_{\mathbb{X}'}. \tag{1.15}$$

For a set $B \subseteq \mathbb{X}'$ let $\partial_{\mathbb{X}'}B$ denote the boundary of B in $\mathbb{X}'$ and $\partial_{\mathbb{X}}B$ the boundary of

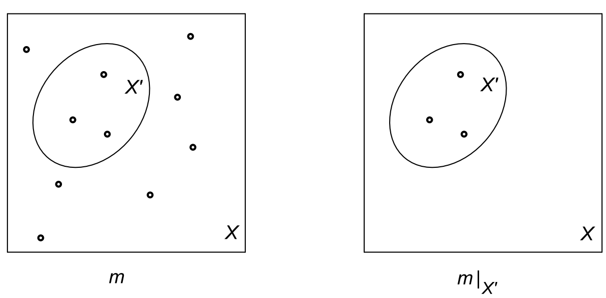

Figure 1.3: *The left figure represents an example of a point process $m \in M_p(X)$ given by its points (or atoms), while the right figure represents m restricted to the subset X'.*

B in $\mathbb{X}$. The restriction map T then posses the following continuity property.

Proposition 1.28. *(Feign et al. [31], Proposition 3.3, (a) and (b)) Let $m \in M_p(\mathbb{X})$ with $m(\partial_{\mathbb{X}}\mathbb{X}') = 0$. Then the following statements hold.*

(a) *The restriction map* $T\colon M_p(\mathbb{X}) \to M_p(\mathbb{X}')$ *defined in (1.15) is continuous at* m, *so if* $m_n \xrightarrow{v} m$ *in* $M_p(\mathbb{X})$, *then* $Tm_n \xrightarrow{v} Tm$ *in* $M_p(\mathbb{X}')$.

(b) *The same conclusion holds if we define* T *the same way but consider it as a mapping* $T\colon M_p(\mathbb{X}) \to M_p(\mathbb{X})$.

At the end of this section we give a result which describes what it means for two Radon point measures m_1 and m_2 to be close. They will be close if in any compact subset of the state space, the finite number of points of m_1 are close in location to the finite number of points of m_2.

Lemma 1.29. *Suppose* m_n, $n \geqslant 0$, *are point measures in* $M_p(\mathbb{X})$ *and* $m_n \xrightarrow{v} m_0$. *For every compact set* $K \subseteq \mathbb{X}$, *such that* $m_0(\partial K) = 0$, *we have for* $n \geqslant n(K)$ *a labeling of the points of* m_n *and* m_0 *in* K *such that*

$$m_n\big|_K = \sum_{i=1}^{s} \delta_{x_i^{(n)}}, \quad m_0\big|_K = \sum_{i=1}^{s} \delta_{x_i^{(0)}},$$

and for every $i = 1, \ldots, s$, *we have in* $\mathbb{X}$,

$$x_i^{(n)} \to x_i^{(0)}, \quad \text{as } n \to \infty.$$

For a proof of this lemma we refer to Lemma 7.1 in Resnick [60] (see also Neveu [53]).

1.5 Weak dependence

The limit theorems in the classical central limit theory were studied under the assumption that the underlying random variables were independent. One of the ways to generalize these theorems is to replace the independence by certain weak dependence conditions. The dependence condition that we shall use will be relatively weak, and it will be implied by the well known strong mixing condition.

In the literature one can find many measures of dependence. Here we shall give five of them which are the most used in applications. For a more detailed discussion about measures of dependence and mixing conditions we refer to Bradley [16]. Suppose $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space. For any σ-field $\mathcal{A} \subseteq \mathcal{F}$, let $L_2(\mathcal{A})$ denote the space of square-integrable, $\mathcal{A}$-measurable, real-valued random variables. For any two σ-fields $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$ define the following set of coefficients which are used to measure dependence:

$$
\begin{aligned}
\alpha(\mathcal{A},\mathcal{B}) &= \sup\{|\mathrm{P}(A\cap B) - \mathrm{P}(A)\mathrm{P}(B)| : A \in \mathcal{A}, B \in \mathcal{B}\}, \\
\phi(\mathcal{A},\mathcal{B}) &= \sup\{|\mathrm{P}(B \,|\, A) - \mathrm{P}(B)| : A \in \mathcal{A}, \mathrm{P}(A) > 0, B \in \mathcal{B}, \}, \\
\psi(\mathcal{A},\mathcal{B}) &= \sup\left\{\frac{|\mathrm{P}(A\cap B) - \mathrm{P}(A)\mathrm{P}(B)|}{\mathrm{P}(A)\mathrm{P}(B)} : A \in \mathcal{A}, B \in \mathcal{B}\right\}, \\
\rho(\mathcal{A},\mathcal{B}) &= \sup\left\{\frac{|\mathrm{E}(XY) - \mathrm{E}X\mathrm{E}Y|}{\sqrt{\mathrm{E}X^2\mathrm{E}Y^2}} : X \in L_2(\mathcal{A}), Y \in L_2(\mathcal{B})\right\}, \\
\beta(\mathcal{A},\mathcal{B}) &= \sup \frac{1}{2}\sum_{i=1}^{I}\sum_{j=1}^{J} |\mathrm{P}(A_i \cap B_j) - \mathrm{P}(A_i)\mathrm{P}(B_j)|,
\end{aligned}
$$

where the supremum in the last equation is taken over all pairs of (finite) partitions $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, B_J\}$ of Ω such that $A_i \in \mathcal{A}$ for all i and $B_j \in \mathcal{B}$ for all j. These coefficients satisfy the following inequalities:

$$
\begin{aligned}
2\alpha(\mathcal{A},\mathcal{B}) &\leqslant \beta(\mathcal{A},\mathcal{B}) \leqslant \phi(\mathcal{A},\mathcal{B}) \leqslant \frac{1}{2}\psi(\mathcal{A},\mathcal{B}); \\
4\alpha(\mathcal{A},\mathcal{B}) &\leqslant \rho(\mathcal{A},\mathcal{B}) \leqslant \psi(\mathcal{A},\mathcal{B}); \\
\rho(\mathcal{A},\mathcal{B}) &\leqslant 2\sqrt{\phi(\mathcal{A},\mathcal{B})}.
\end{aligned}
$$

Now suppose $(X_n)_{n\in\mathbb{Z}}$ is a sequence of random variables. For $-\infty \leqslant k \leqslant l \leqslant \infty$ define $\mathcal{F}_k^l = \sigma(\{X_i : k \leqslant i \leqslant l\})$. Now we are ready to define the mixing conditions for a sequence of random variables.

Definition 1.30. *We say the sequence (X_n) is:*

(i) α-***mixing*** *(or **strongly mixing**) if* $\alpha(n) = \sup_{j\in\mathbb{Z}} \alpha(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}) \to 0$ *as* $n \to \infty$;

(ii) ϕ-***mixing*** *if* $\phi(n) = \sup_{j\in\mathbb{Z}} \phi(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}) \to 0$ *as* $n \to \infty$;

(iii) ψ-***mixing*** *if* $\psi(n) = \sup_{j\in\mathbb{Z}} \psi(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}) \to 0$ *as* $n \to \infty$;

(iv) ρ-***mixing*** *if* $\rho(n) = \sup_{j\in\mathbb{Z}} \rho(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}) \to 0$ *as* $n \to \infty$;

(v) β-***mixing*** *(or **absolutely regular**) if* $\beta(n) = \sup_{j\in\mathbb{Z}} \beta(\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}) \to 0$ *as* $n \to \infty$.

Note that when the sequence (X_n) is strictly stationary, one has simply $\alpha(n) = \alpha(\mathcal{F}_{-\infty}^{0}, \mathcal{F}_{n}^{\infty})$, and the same holds for the other dependence coefficients $\phi(n)$, $\psi(n)$, $\rho(n)$ and $\beta(n)$.

Remark 1.31. By the inequalities above the following implications hold for a given sequence of random variables:

(i) β-mixing $\Rightarrow$ α-mixing;

(ii) ρ-mixing $\Rightarrow$ α-mixing;

(iii) ϕ-mixing $\Rightarrow$ β-mixing and ρ-mixing;

(iv) ψ-mixing $\Rightarrow$ ϕ-mixing.

Aside from transitivity, there are no other implications between these mixing conditions in the general case. However, for some special families of random sequences, e.g. Gaussian sequences and discrete Markov chains, some additional implications hold (see Theorem 3.1, Theorem 3.2 and Theorem 7.1 in Bradley [16]).

Example 1.32. Here, besides examples of random sequences that satisfy some (or all) of these five mixing conditions, we give an example where all of these mixing conditions fail to hold:

(a) A sequence of independent or m-dependent random variables satisfies all of these five mixing conditions.[7]

(b) Suppose $(Z_k)_{k\in\mathbb{Z}}$ is an i.i.d. sequence and the distribution of Z_1 is absolutely continuous with a density which is Gaussian, Cauchy, exponential or uniform (on some interval). Then the random sequence $(X_k)_{k\in\mathbb{Z}}$ defined by

$$X_k = \sum_{j=0}^{\infty} 2^{-j} Z_{k-j},$$

is well defined, strictly stationary and β-mixing (see Example 6.1 in Bradley [15]).

(c) Suppose now $(Z_k)_{k\in\mathbb{Z}}$ is an i.i.d. sequence with $\mathrm{P}(Z_1 = 0) = \mathrm{P}(Z_1 = 1) = 1/2$. Define

$$X_k = \sum_{j=0}^{\infty} 2^{-j+1} Z_{k-j}, \quad k \in \mathbb{Z}.$$

Then the sequence (X_n) is not α-mixing, since $\alpha(n) = 1/4$ for all $n \in \mathbb{N}$ (see Example 6.2 in Bradley [15]). Therefore, in view of Remark 1.31, the remaining four mixing conditions also fail to hold.

Recall $\mathbb{E} = \overline{\mathbb{R}} \setminus \{0\}$, and take a sequence of positive real numbers (a_n) such that $a_n \to \infty$ as $n \to \infty$.

Definition 1.33. *We say a strictly stationary sequence of random variables* (X_n) *satisfies the* ***mixing condition*** $\mathcal{A}'(a_n)$ *if there exist a sequence of positive integers* (r_n) *such that* $r_n \to \infty$ *and* $r_n/n \to 0$ *as* $n \to \infty$*, and such that for every* $f \in C_K^+([0,1]\times\mathbb{E})$*, denoting* $k_n = \lfloor n/r_n \rfloor$*, as* $n \to \infty$*,*

$$\mathrm{E}\exp\Big\{-\sum_{i=1}^{n} f\Big(\frac{i}{n}, \frac{X_i}{a_n}\Big)\Big\} - \prod_{k=1}^{k_n} \mathrm{E}\exp\Big\{-\sum_{i=1}^{r_n} f\Big(\frac{kr_n}{n}, \frac{X_i}{a_n}\Big)\Big\} \to 0. \tag{1.16}$$

[7] The random variables $\{X_n : n \in \mathbb{Z}\}$ are m-dependent if the σ-fields $\mathcal{F}_{j_1}^{k_1}, \dots, \mathcal{F}_{j_l}^{k_l}$ are independent if $k_{i-1} + m < j_i$ for all $i = 2, \dots, l$.

The following result shows that a strictly stationary strongly mixing sequence of regularly varying random variables satisfies the mixing condition $\mathcal{A}'(a_n)$.

Proposition 1.34. *Suppose (X_n) is a strictly stationary sequence of regularly varying random variables with index of regular variation $\alpha > 0$, and (a_n) a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. If (X_n) is strongly mixing then the mixing condition $\mathcal{A}'(a_n)$ holds.*

Proof. Let (l_n) be an arbitrary sequence of positive integers such that $l_n \to \infty$ as $n \to \infty$ and $l_n = o(n^{1/8})$, where $b_n = o(c_n)$ means $b_n/c_n \to 0$ as $n \to \infty$. Define, for any $n \in \mathbb{N}$,

$$r_n = \lfloor \max\{n\sqrt{\alpha_{l_n+1}},\, n^{2/3}\}\rfloor + 1.$$

Then $r_n \to \infty$ as $n \to \infty$. Since the sequence (X_n) is α-mixing, $\alpha_{l_n+1} \to 0$ as $n \to \infty$, and therefore $r_n/n \to 0$ as $n \to \infty$. Put $k_n = \lfloor n/r_n \rfloor$. Then it follows that $k_n \to \infty$ and

$$k_n \alpha_{l_n+1} \to 0 \qquad \text{and} \qquad \frac{k_n l_n}{n} \to 0, \tag{1.17}$$

as $n \to \infty$.

Fix $f \in C_K^+([0,1] \times \mathbb{E})$. We have to show that $I(n) \to 0$ as $n \to \infty$, where

$$I(n) = \bigg| \mathrm{E} \exp\bigg\{ -\sum_{i=1}^{n} f\Big(\frac{i}{n}, \frac{X_i}{a_n}\Big)\bigg\} - \prod_{k=1}^{k_n} \mathrm{E} \exp\bigg\{ -\sum_{i=1}^{r_n} f\Big(\frac{kr_n}{n}, \frac{X_i}{a_n}\Big)\bigg\}\bigg|.$$

We have

$$
\begin{aligned}
I(n) \;\leqslant\;& \Big|\mathrm{E}\exp\Big\{-\sum_{i=1}^{n} f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)\Big\} - \mathrm{E}\exp\Big\{-\sum_{i=1}^{k_n r_n} f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)\Big\}\Big| \\
&+\Big|\mathrm{E}\exp\Big\{-\sum_{i=1}^{k_n r_n} f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)\Big\} - \mathrm{E}\exp\Big\{-\sum_{k=1}^{k_n}\sum_{i=(k-1)r_n+1}^{kr_n-l_n} f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)\Big\}\Big| \\
&+\Big|\mathrm{E}\exp\Big\{-\sum_{k=1}^{k_n}\sum_{i=(k-1)r_n+1}^{kr_n-l_n} f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)\Big\} - \prod_{k=1}^{k_n}\mathrm{E}\exp\Big\{-\sum_{i=1}^{r_n-l_n} f\Big(\frac{kr_n}{n},\frac{X_i}{a_n}\Big)\Big\}\Big| \\
&+\Big|\prod_{k=1}^{k_n}\mathrm{E}\exp\Big\{-\sum_{i=1}^{r_n-l_n} f\Big(\frac{kr_n}{n},\frac{X_i}{a_n}\Big)\Big\} - \prod_{k=1}^{k_n}\mathrm{E}\exp\Big\{-\sum_{i=1}^{r_n} f\Big(\frac{kr_n}{n},\frac{X_i}{a_n}\Big)\Big\}\Big| \\
=:\;& I_1(n)+I_2(n)+I_3(n)+I_4(n)
\end{aligned}
\tag{1.18}
$$

The function f is nonnegative, bounded (by $M>0$ let us suppose) and its support is bounded away from origin, which implies that $f(s,x)=0$ for all $s\in[0,1]$ and $|x|\leqslant\delta$ for some $\delta>0$. Put $j_n=n-k_nr_n$. Then by stationarity and using the inequality $1-e^{-x}\leqslant x$ for any $x\geqslant 0$, we obtain

$$
\begin{aligned}
I_1(n) \;\leqslant\;& \mathrm{E}\Big[\exp\Big\{-\sum_{i=1}^{k_n r_n} f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)\Big\}\cdot\Big|1-\exp\Big\{-\sum_{i=k_nr_n+1}^{n} f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)\Big\}\Big|\Big] \\
\leqslant\;& \mathrm{E}\Big[\sum_{i=k_nr_n+1}^{n} f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)\Big] = \sum_{i=k_nr_n+1}^{n}\mathrm{E}\Big[f\Big(\frac{i}{n},\frac{X_1}{a_n}\Big)1_{\big\{\frac{|X_1|}{a_n}>\delta\big\}}\Big] \\
\leqslant\;& Mj_n\mathrm{P}(|X_1|>\delta a_n).
\end{aligned}
\tag{1.19}
$$

In a similar manner we obtain

$$
I_2(n)\leqslant Mk_nl_n\mathrm{P}(|X_1|>\delta a_n). \tag{1.20}
$$

We have

$$
\begin{aligned}
I_3(n) \quad \leqslant \quad & \Big| \mathrm{E} \exp\Big\{ - \sum_{k=1}^{k_n} \sum_{i=(k-1)r_n+1}^{kr_n - l_n} f\Big(\frac{i}{n}, \frac{X_i}{a_n}\Big)\Big\} \\
& \quad - \mathrm{E} \exp\Big\{ - \sum_{i=1}^{r_n - l_n} f\Big(\frac{i}{n}, \frac{X_i}{a_n}\Big)\Big\} \mathrm{E} \exp\Big\{ - \sum_{k=2}^{k_n} \sum_{i=(k-1)r_n+1}^{kr_n - l_n} f\Big(\frac{i}{n}, \frac{X_i}{a_n}\Big)\Big\}\Big| \\
& + \Big| \mathrm{E} \exp\Big\{ - \sum_{i=1}^{r_n - l_n} f\Big(\frac{i}{n}, \frac{X_i}{a_n}\Big)\Big\} \mathrm{E} \exp\Big\{ - \sum_{k=2}^{k_n} \sum_{i=(k-1)r_n+1}^{kr_n - l_n} f\Big(\frac{i}{n}, \frac{X_i}{a_n}\Big)\Big\} \\
& \quad - \mathrm{E} \exp\Big\{ - \sum_{i=1}^{r_n - l_n} f\Big(\frac{1 \cdot r_n}{n}, \frac{X_i}{a_n}\Big)\Big\} \mathrm{E} \exp\Big\{ - \sum_{k=2}^{k_n} \sum_{i=(k-1)r_n+1}^{kr_n - l_n} f\Big(\frac{i}{n}, \frac{X_i}{a_n}\Big)\Big\}\Big| \\
& + \Big| \mathrm{E} \exp\Big\{ - \sum_{i=1}^{r_n - l_n} f\Big(\frac{1 \cdot r_n}{n}, \frac{X_i}{a_n}\Big)\Big\} \mathrm{E} \exp\Big\{ - \sum_{k=2}^{k_n} \sum_{i=(k-1)r_n+1}^{kr_n - l_n} f\Big(\frac{i}{n}, \frac{X_i}{a_n}\Big)\Big\} \\
& \quad - \prod_{k=1}^{k_n} \mathrm{E} \exp\Big\{ - \sum_{i=1}^{r_n - l_n} f\Big(\frac{kr_n}{n}, \frac{X_i}{a_n}\Big)\Big\}\Big| \\
=: \quad & I_5(n) + I_6(n) + I_7(n).
\end{aligned}
$$

The inequality

$$|\mathrm{E}(gh) - \mathrm{E}g\,\mathrm{E}h| \leqslant 4C_1C_2\alpha_m,$$

for a $\mathcal{F}_{-\infty}^{j}$ measurable function g and a $\mathcal{F}_{j+m}^{\infty}$ measurable function h such that $|g| \leqslant C_1$ and $|h| \leqslant C_2$ (see for instance Lemma 1.2.1 in Lin and Lu [47]), gives

$$I_5(n) \leqslant 4\alpha_{l_n+1}. \tag{1.21}$$

For any $t > 0$ there exists a constant $C(t) > 0$ such that the following inequality holds:

$$|1 - e^{-x}| \leqslant C(t)|x| \quad \text{for all } |x| \leqslant t.$$

This inequality and Lemma 4.3 in Durrett [29] then imply

$$
\begin{aligned}
I_6(n) &\leqslant \mathrm{E}\Big|\exp\Big\{-\sum_{i=1}^{r_n-l_n} f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)\Big\} - \exp\Big\{-\sum_{i=1}^{r_n-l_n} f\Big(\frac{r_n}{n},\frac{X_i}{a_n}\Big)\Big\}\Big| \\
&\leqslant \sum_{i=1}^{r_n-l_n} \mathrm{E}\Big|\exp\Big\{-f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)\Big\} - \exp\Big\{-f\Big(\frac{r_n}{n},\frac{X_i}{a_n}\Big)\Big\}\Big| \\
&\leqslant \sum_{i=1}^{r_n-l_n} \mathrm{E}\Big|1-\exp\Big\{f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big) - f\Big(\frac{r_n}{n},\frac{X_i}{a_n}\Big)\Big\}\Big| \\
&\leqslant C(2M)\sum_{i=1}^{r_n-l_n} \mathrm{E}\Big|f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big) - f\Big(\frac{r_n}{n},\frac{X_i}{a_n}\Big)\Big| \\
&= C(2M)\sum_{i=1}^{r_n-l_n} \mathrm{E}\Big[\Big|f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big) - f\Big(\frac{r_n}{n},\frac{X_i}{a_n}\Big)\Big| 1_{\left\{\frac{|X_i|}{a_n}>\delta\right\}}\Big].
\end{aligned}
$$

Since a continuous function on a compact set is uniformly continuous, it follows that for any $\epsilon > 0$ there exists $\gamma > 0$ such that for $(s,x),(s',x') \in [0,1]\times\{y\in\mathbb{E} : |y|>\delta\}$, if $d_{[0,1]\times\mathbb{E}}((s,x),(s',x')) < \gamma$ then $|f(s,x)-f(s',x')| < \epsilon$, where $d_{[0,1]\times\mathbb{E}}$ denotes the metric on the direct product of metric spaces $[0,1]$ and $\mathbb{E}$, i.e.

$$
d_{[0,1]\times\mathbb{E}}((s,x),(s',x')) = \max\{|s-s'|,\rho(x,x')\},
$$

where ρ is the metric on $\mathbb{E}$ defined in Section 1.1. Since $r_n/n \to 0$ as $n\to\infty$, for large n we have

$$
d_{[0,1]\times\mathbb{E}}\Big(\Big(\frac{i}{n},\frac{X_i}{a_n}\Big),\Big(\frac{r_n}{n},\frac{X_i}{a_n}\Big)\Big) = \frac{|i-r_n|}{n} \leqslant \frac{r_n}{n} < \gamma,
$$

for any $i=1,\dots,r_n-l_n$. Therefore, for large n,

$$
\Big|f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big) - f\Big(\frac{r_n}{n},\frac{X_i}{a_n}\Big)\Big| < \epsilon,
$$

and this implies

$$
I_6(n) \leqslant \epsilon\, C(2M)(r_n-l_n)\mathrm{P}(|X_1|>\delta a_n), \quad \text{for large } n. \tag{1.22}
$$

Taking into account relations (1.21) and (1.22), it follows that, for large n,

$$I_3(n) \leqslant 4\alpha_{l_n+1} + \epsilon\, C(2M) r_n \mathrm{P}(|X_1| > \delta a_n) + I_7(n),$$

and since it is easy to obtain

$$I_7(n) \leqslant \bigg| \mathrm{E} \exp\bigg\{ -\sum_{k=2}^{k_n} \sum_{i=(k-1)r_n+1}^{kr_n - l_n} f\Big(\frac{i}{n}, \frac{X_i}{a_n}\Big)\bigg\} - \prod_{k=2}^{k_n} \mathrm{E} \exp\bigg\{ -\sum_{i=1}^{r_n - l_n} f\Big(\frac{kr_n}{n}, \frac{X_i}{a_n}\Big)\bigg\}\bigg|,$$

we recursively obtain (we repeat the same procedure for $I_7(n)$ as we did for $I_3(n)$ and so on)

$$I_3(n) \leqslant 4k_n\alpha_{l_n+1} + \epsilon\, C(2M) k_n r_n \mathrm{P}(|X_1| > \delta a_n). \tag{1.23}$$

Stationarity and Lemma 4.3 in Durrett [29] imply

$$I_4(n) \leqslant M k_n l_n \mathrm{P}(|X_1| > \delta a_n). \tag{1.24}$$

Thus from relations (1.18), (1.19), (1.20), (1.23) and (1.24) it follows that for large n,

$$\begin{aligned} I(n) \;\; \leqslant & \;\; \Big(M\frac{j_n}{n} + 2M\frac{k_n l_n}{n} + \epsilon\, C(2M)\frac{k_n r_n}{n}\Big) \cdot n\mathrm{P}(|X_1| > a_n) \cdot \frac{\mathrm{P}(|X_1| > \delta a_n)}{\mathrm{P}(|X_1| > a_n)} \\ & + 4k_n \alpha_{l_n+1}. \end{aligned}$$

Since X_1 is regularly varying with index α, by Proposition 1.8 it follows that

$$\frac{\mathrm{P}(|X_1| > \delta a_n)}{\mathrm{P}(|X_1| > a_n)} \to \delta^{-\alpha},$$

as $n \to \infty$. This together with relation (1.17), and the fact that $j_n/n \to 0$, $k_n r_n/n \to 1$ and $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$, imply

$$\limsup_{n\to\infty} I(n) \leqslant \epsilon\, C(2M) \delta^{-\alpha}.$$

But since this holds for all $\epsilon > 0$, we get $\lim_{n\to\infty} I(n) = 0$, and thus condition $\mathcal{A}'(a_n)$ holds. $\square$

1.6 Convergence of point processes under weak dependence

An important ingredient in the proof of our main functional limit theorem in the next chapter will be the convergence in distribution of a sequence of time-space point processes defined by

$$N_n = \sum_{i=1}^{n} \delta_{(i/n, X_i/a_n)} \quad \text{for all } n \in \mathbb{N},$$

where (X_n) is a strictly stationary regularly varying sequence of random variables, and (a_n) is a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$.

Firstly we state the conditions needed for such a convergence. Suppose that the mixing condition $\mathcal{A}'(a_n)$ (see Definition 1.33) holds. To control the dependence of high level exceedances, we introduce the following anti-clustering condition.

Definition 1.35. *We say a strictly stationary sequence of random variables $(X_n)_{n\in\mathbb{Z}}$ satisfies the* ***anti-clustering condition*** *$\mathcal{AC}(a_n)$ if there exists a sequence of positive integers (r_n) such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$, and such that for every $u > 0$,*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathrm{P}\Big(\max_{m \leqslant |i| \leqslant r_n} |X_i| > ua_n \,\Big|\, |X_0| > ua_n \Big) = 0. \tag{1.25}$$

This condition assures that clusters of large values of $|X_n|$ do not last for too long. It was used by Davis and Hsing in [24] in proving that, under the so-called mixing condition $\mathcal{A}(a_n)$ (which we can regard as a version of our condition $\mathcal{A}'(a_n)$ without the time coordinate), the sequence of point processes

$$N_n^* = \sum_{i=1}^{n} \delta_{X_i/a_n}$$

converges in distribution (for details see Theorem 2.7 in [24]).

Put $M_n = \max\{|X_i| : i = 1, \ldots, n\}$, $n \in \mathbb{N}$. In Proposition 4.2 in Basrak and Segers [10], it has been shown that under the anti-clustering condition $\mathcal{AC}(a_n)$ the following value

$$\begin{aligned}\theta &= \lim_{r\to\infty}\lim_{x\to\infty} \mathrm{P}\big(M_r \leqslant x \,\big|\, |X_0| > x\big) \\ &= \mathrm{P}\Big(\sup_{i\geqslant 1}|Y_i| \leqslant 1\Big) = \mathrm{P}\Big(\sup_{i\leqslant -1}|Y_i| \leqslant 1\Big) \end{aligned} \tag{1.26}$$

is strictly positive, where (Y_i) is the tail process of (X_n). Moreover it also holds that $\mathrm{P}\big(\lim_{|n|\to\infty}|Y_n| = 0\big) = 1$, and that for every $u \in (0,\infty)$

$$\theta = \lim_{n\to\infty} \mathrm{P}(M_{r_n} \leq ua_n \mid |X_0| > ua_n) = \lim_{n\to\infty} \frac{\mathrm{P}(M_{r_n} > ua_n)}{r_n \mathrm{P}(|X_0| > ua_n)}. \tag{1.27}$$

The probability that M_{r_n} exceeds ua_n tends to zero as $n \to \infty$.[8] Theorem 4.3 in [10] yields the following weak convergence of a sequence of point processes in the state space $\mathbb{E}$:

$$\left(\sum_{i=1}^{r_n} \delta_{(a_n u)^{-1} X_i} \,\middle|\, M_{r_n} > a_n u\right) \xrightarrow{d} \left(\sum_{n\in\mathbb{Z}} \delta_{Y_n} \,\middle|\, \sup_{i\leqslant -1}|Y_i| \leqslant 1\right). \tag{1.28}$$

Note that as $|Y_n| \to 0$ almost surely as $|n| \to \infty$, the point process $\sum_n \delta_{Y_n}$ is well-defined in $\mathbb{E}$. By (1.26), the probability of the conditioning event on the right-hand side of (1.28) is nonzero. Now we are ready to describe the convergence in distribution of the sequence of point processes (N_n) and to describe the limit. For $u \in (0,\infty)$ let $\mathbb{E}_u = \mathbb{E} \setminus [-u,u]$.

Theorem 1.36. *Suppose (X_n) is a strictly stationary regularly varying random process with index $\alpha > 0$. Assume that it satisfies the mixing conditions $\mathcal{A}'(a_n)$ and the anti-clustering condition $\mathcal{AC}(a_n)$, where (a_n) is a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. Then for every $u \in (0,\infty)$ and as $n \to \infty$,*

$$N_n\big|_{[0,1]\times\mathbb{E}_u} \xrightarrow{d} N^{(u)}\big|_{[0,1]\times\mathbb{E}_u}, \tag{1.29}$$

on $[0,1]\times\mathbb{E}_u$, where $N^{(u)} = \sum_i \sum_j \delta_{(T_i^{(u)}, uZ_{ij})}$ and

[8] Notice that $\mathrm{P}(M_{r_n} > ua_n) \leqslant r_n \mathrm{P}(|X_1| > ua_n) = (r_n/n)\cdot n\mathrm{P}(|X_1| > ua_n) \to 0$ as $n \to \infty$.

1. $\sum_i \delta_{T_i^{(u)}}$ *is a homogeneous Poisson process on* $[0,1]$ *with intensity* $\theta u^{-\alpha}$*;*

2. $(\sum_j \delta_{Z_{ij}})_i$ *is an i.i.d. sequence of point processes in* $\mathbb{E}$*, independent of* $\sum_i \delta_{T_i^{(u)}}$*, and with common distribution equal to the weak limit in* (1.28).

Proof. Let $(X_{k,j})_{j\in\mathbb{N}}$, with $k \in \mathbb{N}$, be independent copies of $(X_j)_{j\in\mathbb{N}}$, and define

$$\hat{N}_n = \sum_{k=1}^{k_n} \hat{N}_{n,k} \qquad \text{with} \qquad \hat{N}_{n,k} = \sum_{j=1}^{r_n} \delta_{(kr_n/n, X_{k,j}/a_n)}.$$

By the mixing condition $\mathcal{A}'(a_n)$ and Theorem 1.23, the weak limits of N_n and $\hat{N}_n$, as $n \to \infty$, must coincide. Take $f \in C_K^+([0,1]\times\mathbb{E}_u)$. Define

$$\widetilde{f}(t,x) = f(t,x)1_{[0,1]\times\mathbb{E}_u}(t,x), \quad (t,x) \in [0,1]\times\mathbb{E}.$$

Then $\widetilde{f} \in C_K^+([0,1]\times\mathbb{E})$. Since $N_n(\widetilde{f}) = \big(N_n\big|_{[0,1]\times\mathbb{E}_u}\big)(f)$ and $\hat{N}_n(\widetilde{f}) = \big(\hat{N}_n\big|_{[0,1]\times\mathbb{E}_u}\big)(f)$, from $\lim_{n\to\infty}\Psi_{N_n}(\widetilde{f}) = \lim_{n\to\infty}\Psi_{\hat{N}_n}(\widetilde{f})$ we get

$$\lim_{n\to\infty} \Psi_{N_n|_{[0,1]\times\mathbb{E}_u}}(f) = \lim_{n\to\infty} \Psi_{\hat{N}_n|_{[0,1]\times\mathbb{E}_u}}(f).$$

Hence, the weak limits of $N_n\big|_{[0,1]\times\mathbb{E}_u}$ and $\hat{N}_n\big|_{[0,1]\times\mathbb{E}_u}$, as $n\to\infty$, also coincide. Therefore to prove (1.29) it is enough to show that the Laplace functional of $\hat{N}_n\big|_{[0,1]\times\mathbb{E}_u}$ converges to the Laplace functional of $N^{(u)}\big|_{[0,1]\times\mathbb{E}_u}$ as $n\to\infty$.

Let $f \in C_K^+([0,1]\times\mathbb{E}_u)$ be arbitrary. Since the function $\widetilde{f}$ is bounded, there exists $M \in (0,\infty)$ such that $0 \leqslant \widetilde{f}(t,x) \leqslant M\,1_{[-u,u]^c}(x)$. Using the inequality $1 - e^{-x} \leqslant x$ for $x \geqslant 0$, we obtain that

$$1 \geqslant \mathrm{E}e^{-\hat{N}_{n,k}(\widetilde{f})} \geqslant \mathrm{E}e^{-M\sum_{i=1}^{r_n} 1_{\{|X_i|>ua_n\}}} \geqslant 1 - Mr_n\mathrm{P}(|X_0| > ua_n).$$

In combination with the elementary bound $0 \leqslant -\log z - (1-z) \leqslant (1-z)^2/z$ for $z \in (0,1]$, it follows that

$$-\log \mathrm{E}e^{-\hat{N}_n(\widetilde{f})} - \sum_{k=1}^{k_n}(1 - \mathrm{E}e^{-\hat{N}_{n,k}(\widetilde{f})})$$

$$
\begin{aligned}
&= -\sum_{k=1}^{k_n} \log \mathrm{E} e^{-\hat{N}_{n,k}(\widetilde{f})} - \sum_{k=1}^{k_n} (1 - \mathrm{E} e^{-\hat{N}_{n,k}(\widetilde{f})}) \\
&\leqslant k_n \frac{[M r_n \mathrm{P}(|X_0| > u a_n)]^2}{1 - M r_n \mathrm{P}(|X_0| > u a_n)} = \frac{M^2}{k_n} \Big(\frac{k_n r_n}{n} \Big)^2 \frac{[n \mathrm{P}(|X_0| > u a_n)]^2}{1 - M r_n \mathrm{P}(|X_0| > u a_n)} \\
&\to 0 \quad \text{as } n \to \infty, \qquad (1.30)
\end{aligned}
$$

since $k_n \to \infty$, $k_n r_n / n \to 1$, $n\mathrm{P}(|X_0| > u a_n) \to u^{-\alpha}$ and $r_n \mathrm{P}(|X_0| > u a_n) \to 0$ as $n \to \infty$. Let T_n be a random variable, uniformly distributed on $\{k r_n / n : k = 1, \ldots, k_n\}$ and independent of $(X_j)_{j \in \mathbb{Z}}$. Then

$$
\begin{aligned}
&\sum_{k=1}^{k_n} (1 - \mathrm{E} e^{-\hat{N}_{n,k}(\widetilde{f})}) \\
&= k_n \mathrm{P}(M_{r_n} > a_n u) \, \frac{1}{k_n} \sum_{k=1}^{k_n} \mathrm{E} \Big[1 - e^{-\sum_{j=1}^{r_n} \widetilde{f}(k r_n / n, X_j / a_n)} \, \Big| \, M_{r_n} > a_n u \Big] \\
&= k_n \mathrm{P}(M_{r_n} > a_n u) \, \frac{1}{k_n} \sum_{k=1}^{k_n} \mathrm{E} \Big[1 - e^{-\sum_{j=1}^{r_n} \widetilde{f}(k r_n / n, X_j / a_n)} \, \Big| \, M_{r_n} > a_n u \Big] \\
&= k_n \mathrm{P}(M_{r_n} > a_n u) \, \mathrm{E} \Big[1 - e^{-\sum_{j=1}^{r_n} \widetilde{f}(T_n, u X_j / (u a_n))} \, \Big| \, M_{r_n} > a_n u \Big] \qquad (1.31)
\end{aligned}
$$

Clearly T_n converge in law to a uniformly distributed random variable T on $(0,1)$. By (1.28) and independence of sequences (T_n) and (X_n)

$$
\bigg(T_n, \sum_{i=1}^{r_n} \delta_{(a_n u)^{-1} X_i} \, \bigg| \, M_{r_n} > a_n u \bigg) \xrightarrow{d} \bigg(T, \sum_{n \in \mathbb{Z}} \delta_{Z_n} \bigg).
$$

where $\sum_n \delta_{Z_n}$ is a point processes on $\mathbb{E}$, independent of the random variable T, and with distribution equal to the weak limit in (1.28). By relation (1.27), for every $u \in (0, \infty)$ it holds that $k_n \mathrm{P}(M_{r_n} > u a_n) \to \theta u^{-\alpha}$ as $n \to \infty$. Thus, from relation (1.31) we have that

$$
\begin{aligned}
\lim_{n \to \infty} \sum_{k=1}^{k_n} (1 - \mathrm{E} e^{-\hat{N}_{n,k}(\widetilde{f})}) &= \theta u^{-\alpha} \mathrm{E} \Big[1 - e^{-\sum_j \widetilde{f}(T, u Z_j)} \Big] \\
&= \int_0^1 \mathrm{E} \Big[1 - e^{-\sum_j \widetilde{f}(t, u Z_j)} \Big] \theta u^{-\alpha} \, dt. \qquad (1.32)
\end{aligned}
$$

Relations (1.30) and (1.32) then yield, as $n \to \infty$,

$$\Psi_{\hat{N}_n}(\widetilde{f}) = \mathrm{E}e^{-\hat{N}_n(\widetilde{f})} \to \exp\Big\{ -\int_0^1 \mathrm{E}[1 - e^{-\sum_j \widetilde{f}(t,uZ_j)}]\theta u^{-\alpha}\,dt\Big\}. \tag{1.33}$$

Define now $g(t) = \mathrm{E}\exp\{-\sum_j \widetilde{f}(t,uZ_j)\}$ for $t \in [0,1]$. Since $\sum_i \delta_{T_i^{(u)}}$ is independent of the i.i.d. sequence $(\sum_j \delta_{Z_{ij}})_i$,

$$\begin{aligned}\mathrm{E}e^{-N^{(u)}(\widetilde{f})} &= \mathrm{E}e^{-\sum_i \sum_j \widetilde{f}(T_i^{(u)},uZ_{ij})} = \mathrm{E}\left(\prod_i \mathrm{E}\left(e^{-\sum_j \widetilde{f}(T_i^{(u)},uZ_{ij})}\,\middle|\,(T_k^{(u)})_k\right)\right)\\ &= \mathrm{E}e^{\sum_i \log g(T_i^{(u)})}.\end{aligned}$$

The right-hand side is the Laplace functional of a homogeneous Poisson process on $[0,1]$ with intensity $\theta u^{-\alpha}$ evaluated in the function $-\log g$. Therefore, it is equal to

$$\exp\Big\{ -\int_0^1 [1-g(t)]\theta u^{-\alpha}\,dt\Big\}$$

(see Example 1.24 (3)). By the definition of g, the integral in the exponent is equal to the one in (1.33). Therefore

$$\Psi_{\hat{N}_n}(\widetilde{f}) \to \Psi_{N^{(u)}}(\widetilde{f}) \quad \text{as } n \to \infty,$$

and this immediately gives

$$\Psi_{\hat{N}_n\big|_{[0,1]\times\mathbb{E}_u}}(f) \to \Psi_{N^{(u)}\big|_{[0,1]\times\mathbb{E}_u}}(f) \quad \text{as } n \to \infty.$$

This completes the proof of the theorem. □

Corollary 1.37. *Assume the setup from Theorem 1.36. Then for every $u \in (0,\infty)$, as $n \to \infty$,*

$$N_n\big|_{[0,1]\times\overline{\mathbb{E}}_u} \xrightarrow{d} N^{(u)}\big|_{[0,1]\times\overline{\mathbb{E}}_u} \tag{1.34}$$

on $[0,1]\times\overline{\mathbb{E}}_u$, where $\overline{\mathbb{E}}_u = [-\infty,-u]\cup[u,\infty]$.

Proof. Let $0 < v < u$. From Theorem 1.36 we have that, as $n \to \infty$,

$$N_n\big|_{[0,1]\times\mathbb{E}_v} \xrightarrow{d} N^{(v)}\big|_{[0,1]\times\mathbb{E}_v} \tag{1.35}$$

on $[0,1] \times \mathbb{E}_v$. Define the restriction map $T\colon M_p([0,1] \times \mathbb{E}_v) \to M_p([0,1] \times \overline{\mathbb{E}}_u)$ by $Tm = m\big|_{[0,1]\times\overline{\mathbb{E}}_u}$. Let

$$\widetilde{\Lambda} = \{m \in M_p([0,1] \times \mathbb{E}_v) : m([0,1] \times \{\pm u\}) = 0\}.$$

By the properties of the tail process, it follows that $\mathrm{P}\big(\sum_j \delta_{vY_j}(\{\pm u\}) = 0\big) = 1$ and therefore, $\mathrm{P}\big(\sum_j \delta_{vZ_{ij}}(\{\pm u\}) = 0\big) = 1$ as well. This implies

$$\mathrm{P}(N^{(v)}\big|_{[0,1]\times\mathbb{E}_v} \in \widetilde{\Lambda}) = 1.$$

Since $\partial_{[0,1]\times\mathbb{E}_v}[0,1]\times\overline{\mathbb{E}}_u = [0,1]\times\{\pm u\}$, for every $m \in \widetilde{\Lambda}$ it holds that $m(\partial_{[0,1]\times\mathbb{E}_v}[0,1]\times\overline{\mathbb{E}}_u) = 0$. Hence, by Proposition 1.28 the restriction map T is continuous on the set $\widetilde{\Lambda}$. Let D_T denote the set of discontinuity points of T. Then

$$\mathrm{P}(N^{(v)}\big|_{[0,1]\times\mathbb{E}_v} \in D_T) \leqslant \mathrm{P}(N^{(v)}\big|_{[0,1]\times\mathbb{E}_v} \notin \widetilde{\Lambda}) = 0.$$

The continuous mapping theorem (see Theorem 3.1 in Resnick [60]) applied to (1.35) yields

$$T(N_n\big|_{[0,1]\times\mathbb{E}_v}) \xrightarrow{d} T(N^{(v)}\big|_{[0,1]\times\mathbb{E}_v}),$$

i.e.

$$N_n\big|_{[0,1]\times\overline{\mathbb{E}}_u} \xrightarrow{d} N^{(v)}\big|_{[0,1]\times\overline{\mathbb{E}}_u} \quad \text{as } n \to \infty, \tag{1.36}$$

on $[0,1] \times \overline{\mathbb{E}}_u$. In a similar manner as above we could show that $N_n\big|_{[0,1]\times\mathbb{E}_u} \xrightarrow{d} N^{(v)}\big|_{[0,1]\times\mathbb{E}_u}$. Recall from Theorem 1.36 that $N_n\big|_{[0,1]\times\mathbb{E}_u} \xrightarrow{d} N^{(u)}\big|_{[0,1]\times\mathbb{E}_u}$. Therefore

$$N^{(v)}\big|_{[0,1]\times\mathbb{E}_u} \stackrel{d}{=} N^{(u)}\big|_{[0,1]\times\mathbb{E}_u}.$$

This together with the fact that $\mathrm{P}(N^{(s)}([0,1] \times \{\pm u\}) = 0) = 1$ for $s = u, v$, suffices to conclude that

$$N^{(v)}\big|_{[0,1]\times\overline{\mathbb{E}}_u} \stackrel{d}{=} N^{(u)}\big|_{[0,1]\times\overline{\mathbb{E}}_u}.$$

Now we can rewrite (1.36) as

$$N_n\big|_{[0,1]\times\overline{\mathbb{E}}_u} \xrightarrow{d} N^{(u)}\big|_{[0,1]\times\overline{\mathbb{E}}_u}.$$

□

Corollary 1.38. *With the assumptions as in Corollary 1.37 it holds that, as $n \to \infty$,*

$$N_n\big|_{[0,1]\times\mathbb{E}_u} \xrightarrow{d} N^{(u)}\big|_{[0,1]\times\mathbb{E}_u}$$

on $[0,1] \times \overline{\mathbb{E}}_u$.

Note that the only difference with Theorem 1.36 is that here we have the convergence on the state space $[0,1] \times \overline{\mathbb{E}}_u$, while the convergence in Theorem 1.36 is on $[0,1] \times \mathbb{E}_u$.

Proof of Corollary 1.38. Define $T_1 \colon M_p([0,1]\times\overline{\mathbb{E}}_u) \to M_p([0,1]\times\overline{\mathbb{E}}_u)$ by $T_1 m = m\big|_{[0,1]\times\mathbb{E}_u}$. Let

$$\widetilde{\Lambda}_1 = \{m \in M_p([0,1] \times \overline{\mathbb{E}}_u) : m([0,1] \times \{\pm u\}) = 0\}.$$

Then in a similar way as in the proof of Corollary 1.37 we obtain that

$$\mathrm{P}(N^{(u)}\big|_{[0,1]\times\overline{\mathbb{E}}_u} \in \widetilde{\Lambda}_1) = 1,$$

and that the functional T_1 is continuous on the set $\widetilde{\Lambda}_1$. Therefore from relation (1.34) and the continuous mapping theorem it follows that, as $n \to \infty$,

$$T_1(N_n\big|_{[0,1]\times\overline{\mathbb{E}}_u}) \xrightarrow{d} T_1(N^{(u)}\big|_{[0,1]\times\overline{\mathbb{E}}_u}),$$

i.e.

$$N_n\big|_{[0,1]\times\mathbb{E}_u} \xrightarrow{d} N^{(u)}\big|_{[0,1]\times\mathbb{E}_u}$$

on $[0,1]\times\overline{\mathbb{E}}_u$. $\square$

1.7 Lévy processes

The limit processes in functional limit theorems which will be studied in the forthcoming chapters will belong to a special class of Lévy processes. In this section we introduce a framework of the theory of Lévy processes needed to describe these limit processes.

It will be very useful to connect Lévy processes with infinitely divisible distributions and their Lévy-Khintchine representations. For a textbook treatment of Lévy processes we refer to Bertoin [11], Kyprianou [44], Samorodnitsky and Taqqu [62] and Sato [63].

The probability measure μ on $\mathbb{R}$ is *infinitely divisible* if for every $n\in\mathbb{N}$ there is a probability measure μ_n such that $\mu=\mu_n^{n*}$, where μ_n^{n*} denotes the n-fold convolution of μ_n.[9]

A random variable X is said to has an *infinitely divisible distribution* if its probability distribution P_X is infinitely divisible.[10] Equivalently X has an infinitely divisible distribution if for every $n\in\mathbb{N}$ there exists a sequence of i.i.d. random variables $X_{1,n},\ldots,X_{n,n}$ such that

$$X \stackrel{d}{=} X_{1,n}+\ldots+X_{n,n},$$

where $\stackrel{d}{=}$ denotes equality in distribution.

The following theorem gives a representation of characteristic functions of infinitely divisible distributions, and it is called the *Lévy-Khintchine representation.* Let $\widehat{\mu}$ denote

[9] The convolution $\nu_1*\nu_2$ of two distributions ν_1 and ν_2 on $\mathbb{R}$ is a distribution defined by $(\nu_1*\nu_2)(B)=\iint_{\mathbb{R}\times\mathbb{R}}1_B(x+y)\nu_1(dx)\nu_2(dy)$, $B\in\mathcal{B}(\mathbb{R}^2)$.

[10] The probability distribution (or probability law) of a random variable X is the probability measure P_X defined by $\mathrm{P}_X(B)=\mathrm{P}(X\in B)$, $B\in\mathcal{B}(\mathbb{R})$.

the characteristic function of a probability measure μ, i.e.

$$\widehat{\mu}(z) = \int_{\mathbb{R}} e^{izx}\mu(dx), \quad z \in \mathbb{R}.$$

Then restating the statement of Theorem 8.1 in Sato [63] in the 1-dimensional case we obtain the following theorem.

Theorem 1.39. *(i) If μ is an infinitely divisible distribution on $\mathbb{R}$, then*

$$\widehat{\mu}(z) = \exp\Big[-\frac{1}{2}az^2 + ibz + \int_{\mathbb{R}} (e^{izx} - 1 - izx1_{\{|x|\leqslant 1\}})\nu(dx)\Big], \quad z \in \mathbb{R}, \quad (1.37)$$

where $a \geqslant 0$, $b \in \mathbb{R}$ and ν is a measure on $\mathbb{R}$ satisfying

$$\nu(\{0\}) = 0 \qquad \text{and} \qquad \int_{\mathbb{R}} (|x|^2 \wedge 1)\nu(dx) < \infty. \qquad (1.38)$$

(ii) The representation of $\widehat{\mu}(z)$ in (i) by a, b and ν is unique.

(iii) Conversely, if $a \geqslant 0$, $b \in \mathbb{R}$ and ν is a measure satisfying (1.38), then there exists an infinitely divisible distribution μ whose characteristic function is given by (1.37).

The triple (a, ν, b) in Theorem 1.39 is called the *characteristic triple* of μ, and the measure ν is called the *Lévy measure* of μ. Now we turn to the definition of Lévy processes.

Definition 1.40. *A stochastic process $X = \{X_t : t \geqslant 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is a* ***Lévy process*** *if it possesses the following properties.*

(1) For $0 \leqslant s \leqslant t$, $X_t - X_s$ is independent of $\{X_u : u \leqslant s\}$.

(2) For $0 \leqslant s \leqslant t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

(3) $\mathrm{P}(X_0 = 0) = 1$.

(4) The paths of X are almost surely right continuous with left limits.

Remark 1.41. If $X = \{X_t : t \geqslant 0\}$ is a Lévy process, then it is not hard to see that for every $t > 0$ the random variable X_t has an infinitely divisible distribution. Indeed write X_t in the form

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \ldots + (X_t - X_{(n-1)t/n}),$$

for an arbitrary $n \in \mathbb{N}$. On the right hand side, by properties (1) and (2) in Definition 1.40, we have the sum of n i.i.d. random variables, which shows that X_t has an infinitely divisible distribution.

In particular, if $X = \{X_t : t \geqslant 0\}$ is a Lévy process, then the random variable X_1 has an infinitely divisible distribution, i.e. P_{X_1} is infinitely divisible. Conversely, if μ is an infinitely divisible probability measure on $\mathbb{R}$, then there exists a Lévy process $X = \{X_t : t \geqslant 0\}$ such that $\mu = \mathrm{P}_{X_1}$ (see for instance Theorem 13.12 in Kallenberg [40]). Hence the following result holds.

Theorem 1.42. *For a probability measure μ on $\mathbb{R}$, these conditions are equivalent:*

(i) μ is infinitely divisible;

(ii) $\mu = \mathrm{P}_{X_1}$ for some Lévy process $X = \{X_t : t \geqslant 0\}$.

Under these conditions, the distribution of X is determined by μ.

Remark 1.43. For a Lévy process $X = \{X_t : t \geqslant 0\}$ let

$$\varphi_{X_t}(z) = \mathrm{E}[e^{izX_t}], \quad z \in \mathbb{R},$$

denote the characteristic function of the random variable X_t. Then using the properties of Lévy processes it follows that

$$\varphi_{X_t}(z) = [\varphi_{X_1}(z)]^t,$$

for any $t \geqslant 0$ (see Kyprianou [44, p. 4]).

Having in mind the relation between infinitely divisible distributions and Lévy processes expressed in Theorem 1.42, for a characteristic triple (a, ν, b) of a probability distribution P_{X_1} we say it is also the *characteristic triple* of a Lévy process $X = \{X_t : t \geqslant 0\}$.

Next we turn our attention to a special class of Lévy processes, i.e. the stable Lévy processes. Firstly, for a random variable Y we say it has a *stable distribution* if for every $n \in \mathbb{N}$ there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$Y_1 + \ldots + Y_n \stackrel{d}{=} a_n Y + b_n, \tag{1.39}$$

where $Y_1, \ldots, Y_n$ are independent copies of Y. Relation (1.39) can be rewritten in the form

$$\left(Y_1 - \frac{b_n}{n}\right) + \ldots + \left(Y_n - \frac{b_n}{n}\right) \stackrel{d}{=} a_n Y.$$

Therefore, $a_n Y$ is infinitely divisible. From this we immediately obtain that Y is also infinitely divisible. Thus every stable random variable is infinitely divisible. It turns out that in (1.39) we necessarily have $a_n = n^{1/\alpha}$ for $\alpha \in (0, 2]$ (see Theorem 1 in Feller [32, p. 166]). The number α is called the *index of stability* (or *characteristic exponent*). A stable random variable with index of stability α is called *α-stable*.

Since stable random variables are infinitely divisible, their characteristic functions have the form given in (1.37). The following result gives the characterization of stable distributions in terms of theirs characteristic triples (for a proof see Sato [63], Theorem 14.3 (ii)).

Theorem 1.44. *Let Y be a non-degenerate infinitely divisible random variable with characteristic triple (a, ν, b). Let $0 < \alpha < 2$. Then Y is α-stable if and only if $a = 0$*

and

$$\nu(uB) = u^{-\alpha}\nu(B) \quad \text{for all } u > 0,\ B \in \mathcal{B}(\mathbb{R}).$$

Sometimes is convenient to rewrite the characteristic function of a stable random variable in the form given in the next result (see for instance Sato [63], Theorem 14.15).

Theorem 1.45. *Let $\alpha \in (0, 2]$. If Y is a non-degenerate α-stable random variable, then its characteristic function φ_Y is of the form*

$$\varphi_Y(z) = \exp\Big[-c|z|^\alpha\Big(1 - i\beta(\operatorname{sign} z)\tan\frac{\pi\alpha}{2}\Big) + i\tau z\Big] \quad \text{for } \alpha \neq 1, \tag{1.40}$$

$$\varphi_Y(z) = \exp\Big[-c|z|\Big(1 + i\beta\frac{2}{\pi}(\operatorname{sign} z)\log|z|\Big) + i\tau z\Big] \quad \text{for } \alpha = 1, \tag{1.41}$$

with $c > 0$, $\beta \in [-1, 1]$ and $\tau \in \mathbb{R}$. Here c, β and τ are uniquely determined by Y.[11] *Conversely, for every $c > 0$, $\beta \in [-1, 1]$ and $\tau \in \mathbb{R}$, there is a non-degenerate α-stable random variable Y satisfying (1.40) or (1.41).*

Example 1.46. When $\alpha = 2$, the characteristic function in (1.40) becomes $\varphi_Y(z) = \exp\{-cz^2 + i\tau z\}$. This is the characteristic function of a Gaussian random variable with mean τ and variance $2c$.

Remark 1.47. The representations of the characteristic function of a stable distribution in the Lévy-Khintchine representation (1.37) and relations (1.40) and (1.41) are connected in the following way. From Theorem 1.45 we know that the characteristic function of a non-degenerate stable random variable is characterized by four parameters

$$\alpha \in [0, 2],\ c > 0,\ \beta \in [-1, 1],\ \tau \in \mathbb{R}.$$

The characteristic triple (a, ν, b) of an α-stable random variable for $\alpha \in (0, 2)$ is then given by

$$a = 0, \quad \nu(dx) = \big(c_1 1_{(0,\infty)}(x) + c_2 1_{(-\infty,0)}(x)\big)\,|x|^{-1-\alpha}dx \quad \text{and} \quad b = \tau - d,$$

[11] β is irrelevant when $\alpha = 2$, and in this case we take $\beta = 0$.

where

(1) $\alpha \in (0,1)$: $c_1 = \frac{-c(1+\beta)}{2\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}}$, $c_2 = \frac{-c(1-\beta)}{2\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}}$, $d = -\int_{|x|\leqslant 1} x\,\nu(dx)$;

(2) $\alpha = 1$: $c_1 = \frac{c(1+\beta)}{\pi}$, $c_2 = \frac{c(1-\beta)}{\pi}$, $d = (c_1 - c_2)\big(\int_1^\infty \frac{\sin r}{r^2}\,dr + \int_0^1 \frac{\sin r - r}{r^2}\,dr\big)$;

(3) $\alpha \in (1,2)$: $c_1 = \frac{-c(1+\beta)}{2\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}}$, $c_2 = \frac{-c(1-\beta)}{2\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}}$, $d = \int_{|x|>1} x\,\nu(dx)$

(see Lemma 2 in Feller [32, p. 541] and the computations in Sato [63, p. 84, 85]). The characteristic triple of a 2-stable random variable is of the form $(2c, 0, \tau)$.

Definition 1.48. *A Lévy process $X = \{X_t : t \geqslant 0\}$ is called α-**stable** if the random variable X_1 is α-stable.*

Remark 1.49. If in relation (1.39) we have $b_n = 0$, then the random variable Y is said to have a *strictly stable distribution*. The characteristic function of a strictly α-stable random variable with $\alpha \neq 1$, is given by relation (1.40) with $\tau = 0$, while the characteristic function of a strictly 1-stable random variable is given by relation (1.41) with $\beta = 0$ (see Property 1.2.6 and Property 1.2.8 in Samorodnitsky and Taqqu [62]).

Chapter 2

Functional limit theorem with M_1 convergence

In this chapter we prove the main result of this thesis, namely the functional limit theorem for regularly varying random processes under weak dependence, the anti-clustering condition $\mathcal{AC}(a_n)$, a condition on the tail process and an additional technical condition for the case when $\alpha \in [1, 2)$, where α is the index of regular variation of the random process. The convergence in this theorem will be given with respect to Skorohod's M_1 topology.

2.1 Space $D[0, 1]$ and Skorohod's J_1 and M_1 metrics

Since the stochastic process that we are going to study have discontinuities, for the underlying function space of sample paths of the stochastic processes we choose the space $D[0, 1]$ of all right-continuous real valued functions with left limits defined on $[0, 1]$. The space $D[0, 1]$ is also known as the space of *càdlàg* functions.[1] The space $C[0, 1]$ of all continuous real valued functions on $[0, 1]$ is clearly a subset of $D[0, 1]$. The well known and mostly used metric on $C[0, 1]$ is the *uniform metric* defined by

$$d(x, y) = \|x - y\|_{[0,1]} = \sup_{t \in [0,1]} |x(t) - y(t)|, \quad x, y \in C[0, 1].$$

[1]Càdlàg is an acronym for the French *continue à droite, limites à gauche.*

While this metric works well on $C[0,1]$, it causes problems on $D[0,1]$.

Example 2.1. Define

$$x_n(t) = 1_{[0,\frac{1}{2}+\frac{1}{n})}(t), \qquad x(t) = 1_{[0,\frac{1}{2})}(t),$$

for $n \geqslant 3$ and $t \in [0,1]$. Then for every $n \geqslant 3$,

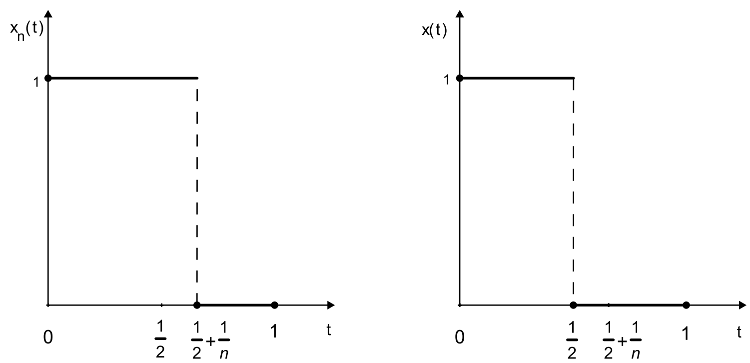

Figure 2.1: *Plots of the functions x_n and x.*

$$d(x_n, x) \geqslant \left| x_n\Big(\frac{1}{2} + \frac{1}{2n}\Big) - x\Big(\frac{1}{2} + \frac{1}{2n}\Big) \right| = 1,$$

which implies that the sequence (x_n) does not converge to x in the uniform metric. We want to have a metric in which (x_n) converges to x.

The uniform metric allows uniformly small perturbations of the space coordinate, but not of the time coordinate. So we need a new metric which allows also small perturbations of the time scale. The metric we are looking for was introduced by Skorohod [64] and is defined in the following way. Let Δ be the set of strictly increasing continuous functions $\lambda\colon [0,1] \to [0,1]$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$, and let $e \in \Delta$ be the identity map on $[0,1]$, i.e. $e(t) = t$ for all $t \in [0,1]$. For $x, y \in D[0,1]$ define

$$d_{J_1}(x,y) = \inf\{\|x \circ \lambda - y\|_{[0,1]} \vee \|\lambda - e\|_{[0,1]} : \lambda \in \Delta\},$$

where $s \vee t$ denotes $\max\{s,t\}$. Then d_{J_1} is a metric on $D[0,1]$ (see Billingsley [12, p. 111]) and is called the (*Skorohod*) J_1 *metric*. Some simple properties of the metric d_{J_1} are collected in the following proposition (for a proof see Resnick [60, p. 47]).

Proposition 2.2. *The following statements hold.*

1. *For a sequence of functions $(x_n)_{n \geqslant 0}$ in $D[0,1]$ it holds that $d_{J_1}(x_n, x_0) \to 0$ if and only if there exists a sequence of functions (λ_n) in Δ such that, as $n \to \infty$,*
$$\|\lambda_n - e\|_{[0,1]} \to 0 \quad \textit{and} \quad \|x_n \circ \lambda_n - x_0\|_{[0,1]} \to 0.$$
2. *$d_{J_1}(x,y) \leqslant \|x-y\|_{[0,1]}$ for all $x, y \in D[0,1]$.*
3. *If $d_{J_1}(x_n, x_0) \to 0$ as $n \to \infty$, for $x_n \in D[0,1], n \geqslant 0$, then for every continuity point $t \in [0,1]$ of x_0 it holds that $x_n(t) \to x_0(t)$ as $n \to \infty$.*
4. *If $d_{J_1}(x_n, x_0) \to \infty$ as $n \to \infty$ and $x_0 \in C[0,1]$, then $\|x_n - x_0\|_{[0,1]} \to 0$.*

The space $D[0,1]$ endowed with the J_1 metric is a separable metric space, but it is not complete since the sequence (x_n) defined by
$$x_n(t) = 1_{[\frac{1}{2}, \frac{1}{2}+\frac{1}{n})}(t)$$
is a Cauchy sequence in the metric d_{J_1}, but it is not convergent. For details we refer to Billingsley [12], where is also given a metric, topologically equivalent to d_{J_1} (i.e. gives the same topology as d_{J_1}), under which $D[0,1]$ is complete.

Example 2.3. Recall the functions defined in Example 2.1. For every $n \geqslant 3$ put
$$\lambda_n(t) = \begin{cases} 2\big(\frac{1}{2} + \frac{1}{n}\big)t, & t \in [0, \frac{1}{2}), \\ \big(1 - \frac{2}{n}\big)t + \frac{2}{n}, & t \in [\frac{1}{2}, 1]. \end{cases}$$
Then $\lambda_n \in \Delta$, and since $\|\lambda_n - e\|_{[0,1]} = n^{-1}$ and $\|x_n \circ \lambda_n - x\|_{[0,1]} = 0$, by the first statement in Proposition 2.2 it follows that $d_{J_1}(x_n, x) \to 0$ as $n \to \infty$. Therefore the sequence (x_n) converges to x in the J_1 metric.

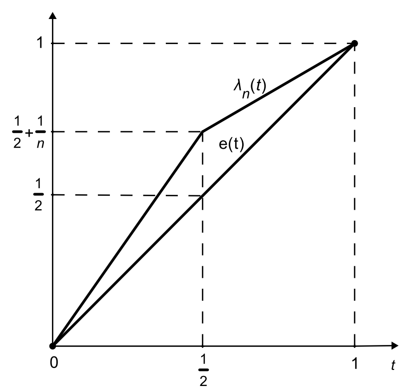

Figure 2.2: *Plots of the functions λ_n and e.*

The J_1 metric will be useful in the next chapter when we shall study the i.i.d. case and the isolated extremes case. Roughly speaking, in these cases any jump in the limiting process will be approached by one jump in the converging sequence. But when the limiting jump is approached in more then one jump in the converging sequence, then the J_1 metric does not work well, and we have to find a more suitable metric.

For $x \in D[0,1]$ the *completed graph* of x is the set

$$\Gamma_x = \{(t,z) \in [0,1] \times \mathbb{R} : z = \lambda x(t-) + (1-\lambda)x(t) \text{ for some } \lambda \in [0,1]\},$$

where $x(t-)$ is the left limit of x at t. Besides the points of the graph $\{(t, x(t)) : t \in [0,1]\}$, the completed graph of x also contains the vertical line segments joining $(t, x(t))$ and $(t, x(t-))$ for all discontinuity points t of x. We define an *order* on the graph Γ_x by saying that $(t_1, z_1) \leqslant (t_2, z_2)$ if either (i) $t_1 < t_2$ or (ii) $t_1 = t_2$ and $|x(t_1-) - z_1| \leqslant |x(t_2-) - z_2|$. A *parametric representation* of the completed graph Γ_x is a continuous nondecreasing function (r, u) mapping $[0,1]$ onto Γ_x, with r being the time component and u being the spatial component. Let $\Pi(x)$ denote the set of parametric representations of the graph Γ_x. For $x_1, x_2 \in D[0,1]$ define

$$d_{M_1}(x_1, x_2) = \inf\{\|r_1 - r_2\|_{[0,1]} \vee \|u_1 - u_2\|_{[0,1]} : (r_i, u_i) \in \Pi(x_i), i = 1,2\}.$$

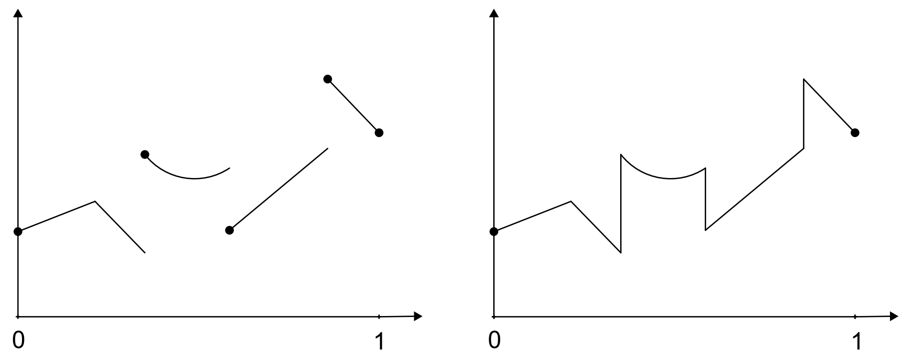

Figure 2.3: *A function in* $D[0,1]$ *and its completed graph.*

Then d_{M_1} is a metric on $D[0,1]$ (see Theorem 12.3.1 in Whitt [69]) and is called the (*Skorohod*) M_1 *metric.*

The J_1 and M_1 metrics are related by the following inequality

$$d_{M_1}(x,y) \leqslant d_{J_1}(x,y), \quad x,y \in D[0,1] \tag{2.1}$$

(see for instance Theorem 6.3.2 in Whitt [68]). The main properties of the M_1 metric which we shall use later on are given in the following two results (for a proof see Corollary 12.5.1 and Corollary 12.7.1 in Whitt [69]).

Proposition 2.4. *If* $x_n \in D[0,1]$ *is a monotone function for each* $n \in \mathbb{N}$*, then* $d_{M_1}(x_n, x) \to 0$ *for* $x \in D[0,1]$ *if and only if* $x_n(t) \to x(t)$ *for all* t *in a dense subset of* $[0,1]$ *including* 0 *and* 1.

Let D_x denote the set of discontinuities of $x \in D[0,1]$, i.e.

$$D_x = \{t \in (0,1] : x(t-) \neq x(t)\}.$$

Proposition 2.5. *Let* $x, y, x_n, y_n \in D[0,1]$, $n \in \mathbb{N}$. *If* $d_{M_1}(x_n, x) \to 0$ *and* $d_{M_1}(y_n, y) \to 0$ *as* $n \to \infty$*, and*

$$D_x \cap D_y = \emptyset,$$

then $d_{M_1}(x_n + y_n, x + y) \to 0$ *as* $n \to \infty$.

Example 2.6. Define

$$y_n(t) = \frac{1}{2} 1_{[\frac{1}{2}-\frac{1}{n},\frac{1}{2})}(t) + 1_{[\frac{1}{2},1]}(t), \quad y(t) = 1_{[\frac{1}{2},1]}(t),$$

for $n \geqslant 3$ and $t \in [0,1]$. Then little calculations yield $d_{J_1}(y_n, y) \geqslant 1/2$ for every $n \geqslant 3$, showing that the sequence (y_n) does not converge to y in the J_1 metric. But things change if we use the M_1 metric. For the following parametric representations (r, u) of

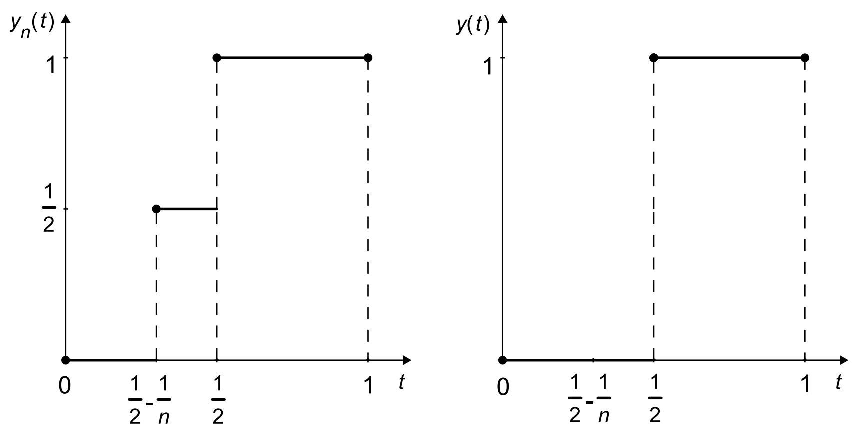

Figure 2.4: *Plots of the functions y_n and y.*

Γ_y and (r_n, u_n) of Γ_{y_n}, defined by

$$\begin{aligned}
r(s) &= \frac{5s}{2} 1_{[0,\frac{1}{5}]}(s) + \frac{1}{2} 1_{(\frac{1}{5},\frac{4}{5}]}(s) + \frac{5s-3}{2} 1_{(\frac{4}{5},1]}(s), \\
r_n(s) &= 5s\Big(\frac{1}{2} - \frac{1}{n}\Big) 1_{[0,\frac{1}{5}]}(s) + \Big(\frac{1}{2} - \frac{1}{n}\Big) 1_{(\frac{1}{5},\frac{2}{5}]}(s) + \Big(\frac{5s-3}{n} + \frac{1}{2}\Big) 1_{(\frac{2}{5},\frac{3}{5}]}(s) \\
&\quad + \frac{1}{2} 1_{(\frac{3}{5},\frac{4}{5}]}(s) + \frac{5s-3}{2} 1_{(\frac{4}{5},1]}(s), \\
u(s) &= u_n(s) = \frac{5s-1}{2} 1_{(\frac{1}{5},\frac{2}{5}]}(s) + \frac{1}{2} 1_{(\frac{2}{5},\frac{3}{5}]}(s) + \frac{5s-2}{2} 1_{(\frac{3}{5},\frac{4}{5}]}(s) + 1_{(\frac{4}{5},1]}(s),
\end{aligned}$$

we have $\|r_n - r\|_{[0,1]} = n^{-1}$ and $\|u_n - u\|_{[0,1]} = 0$. Therefore $d_{M_1}(y_n, y) \to 0$ as $n \to \infty$.

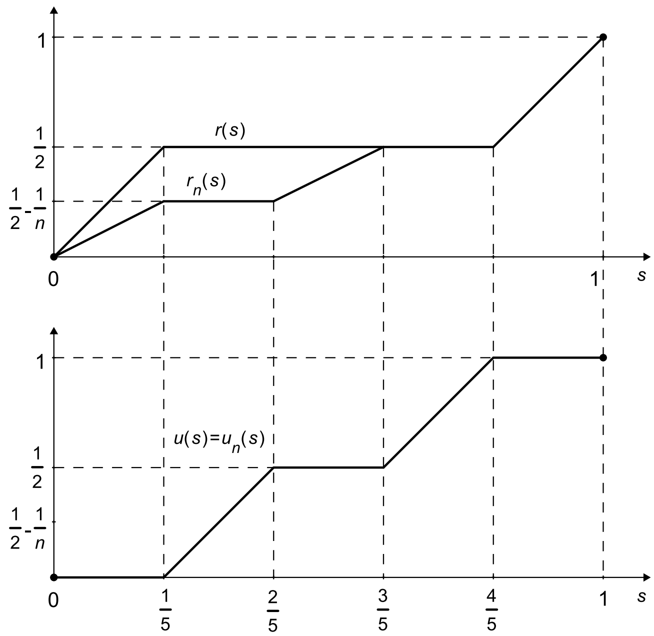

Figure 2.5: *Plots of parametric representations* (r, u) *of* Γ_y *and* (r_n, u_n) *of* Γ_{y_n}.

Remark 2.7. The M_1 topology (induced by Skorohod's M_1 metric) was introduced in Skorohod [64], along with J_1, J_2 and M_2 topologies. Topology J_1 is the most used Skorohod's topology. But in this thesis the M_1 topology will be used in most of considerations. It is straightforward to see that all these topologies are weaker then the uniform topology, but stronger then the L_p topologies on $D[0,1]$ induced by the norms

$$\|x\|_p = \Big(\int_0^1 |x(t)|^p \, dt \Big)^{1/p},$$

for $p \geqslant 1$. The last statement follows from the fact that convergence in all four of Skorohod's topologies implies pointwise convergence on the set of all continuity points of the limiting function, and since this set contains all except at most countably many points from $[0,1]$, we obtain convergence in L_p topology.

For a more detailed treatment of the space $D[0,1]$ and Skorohod's topologies we refer to Whitt [69].

2.2 Summation functional

Fix an arbitrary $u > 0$. The proof of our main theorem depends on the continuity properties of the summation functional

$$\psi^{(u)} \colon M_p([0,1] \times \overline{\mathbb{E}}_u) \to D[0,1]$$

defined by

$$\psi^{(u)}\Big(\sum_i \delta_{(t_i,x_i)}\Big)(t) = \sum_{t_i \le t} x_i \, 1_{\{u<|x_i|<\infty\}}, \qquad t \in [0,1].$$

Observe that $\psi^{(u)}$ is well defined because $[0,1] \times \overline{\mathbb{E}}_u$ is a compact set. In the sequel the space $M_p([0,1] \times \overline{\mathbb{E}}_u)$ is equipped with the vague topology and $D[0,1]$ is equipped with the M_1 topology.

The summation functional $\psi^{(u)}$ is not continuous on the set $[0,1] \times \overline{\mathbb{E}}_u$ (see Example 2.9 below), but we will show that it is continuous on the set $\Lambda = \Lambda_1 \cap \Lambda_2$, where

$$\Lambda_1 = \{\eta \in M_p([0,1] \times \overline{\mathbb{E}}_u) \,:\, \eta(\{0,1\} \times \mathbb{E}_u) = \eta([0,1] \times \{\pm\infty, \pm u\}) = 0\},$$

$$\Lambda_2 = \{\eta \in M_p([0,1] \times \overline{\mathbb{E}}_u) \,:\, \eta(\{t\} \times [u,\infty]) \cdot \eta(\{t\} \times [-\infty,-u]) = 0 \text{ for all } t \in [0,1]\}.$$

Observe that the elements of Λ_2 have the property that atoms with the same time coordinate are all on the same side of the space axis.

Lemma 2.8. *The summation functional $\psi^{(u)} \colon M_p([0,1] \times \overline{\mathbb{E}}_u) \to D[0,1]$ is continuous on the set Λ, when $D[0,1]$ is endowed with Skorohod's M_1 topology.*

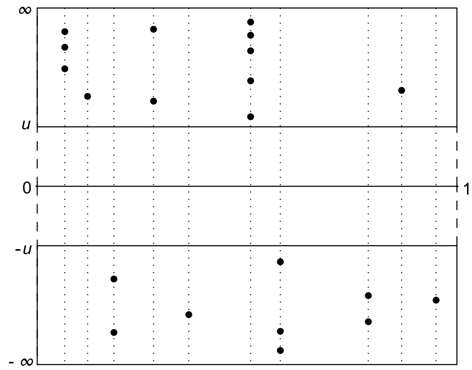

Figure 2.6: *An example of a point process belonging to the set* Λ*.*

Proof. Take an arbitrary $\eta \in \Lambda$ and suppose that $\eta_n \xrightarrow{v} \eta$ in $M_p([0,1] \times \overline{\mathbb{E}}_u)$ as $n \to \infty$. We will show that $\psi^{(u)}(\eta_n) \to \psi^{(u)}(\eta)$ in $D[0,1]$ according to the M_1 topology. By Proposition 2.4, M_1 convergence for monotone functions amounts to pointwise convergence in a dense subset of points plus convergence at the endpoints. Our proof is based on an extension of this criterion to piecewise monotone functions. This cut-and-paste approach is justified in view of Lemma 12.9.2 in Whitt [69], provided that the limit function is continuous at the cutting points.

Since the set $[0,1] \times \overline{\mathbb{E}}_u$ is compact, there exists a nonnegative integer $k = k(\eta)$ such that

$$\eta([0,1] \times \overline{\mathbb{E}}_u) = k < \infty.$$

By assumptions, η does not have any atoms on the border of the set $[0,1] \times \overline{\mathbb{E}}_u$. As a consequence ofLemma 1.29 there exists a positive integer n_0 such that for all $n \geqslant n_0$ it holds that

$$\eta_n([0,1] \times \overline{\mathbb{E}}_u) = k.$$

If $k = 0$, there is nothing to prove, so assume $k \geqslant 1$ and let (t_i, x_i) for $i = 1, \ldots, k$, be the atoms of η in $[0,1] \times \overline{\mathbb{E}}_u$. By the same lemma, the k atoms $(t_i^{(n)}, x_i^{(n)})$ of η_n in

$[0,1] \times \overline{\mathbb{E}}_u$ (for $n \geqslant n_0$) can be labeled in such a way that for every $i = 1, \ldots, k$ we have

$$(t_i^{(n)}, x_i^{(n)}) \to (t_i, x_i), \qquad \text{as } n \to \infty.$$

In particular, for any $\delta > 0$ we can find a positive integer n_δ such that for all $n \geqslant n_\delta$,

$$\begin{aligned} \eta_n([0,1] \times \overline{\mathbb{E}}_u) &= k, \\ |t_i^{(n)} - t_i| < \delta \quad \text{and} \quad |x_i^{(n)} - x_i| &< \delta, \quad \text{for } i = 1, \ldots, k. \end{aligned} \tag{2.2}$$

Let the sequence

$$0 < \tau_1 < \tau_2 < \ldots < \tau_p < 1$$

be such that the sets $\{\tau_1, \ldots, \tau_p\}$ and $\{t_1, \ldots, t_k\}$ coincide. Since η can have several atoms with the same time coordinate, it always holds that $p \leqslant k$. Put $\tau_0 = 0$, $\tau_{p+1} = 1$ and take

$$0 < r < \frac{1}{2} \min_{0 \leqslant i \leqslant p} |\tau_{i+1} - \tau_i|.$$

For any $t \in [0,1] \setminus \{\tau_1, \ldots, \tau_p\}$ we can find $\delta \in (0, u)$ such that

$$\delta < r \quad \text{and} \quad \delta < \min_{1 \leqslant i \leqslant p} |t - \tau_i|.$$

Then relation (2.2), for $n \geqslant n_\delta$, implies that $t_i^{(n)} \leqslant t$ is equivalent to $t_i \leqslant t$, and we obtain

$$|\psi^{(u)}(\eta_n)(t) - \psi^{(u)}(\eta)(t)| = \Bigg| \sum_{t_i^{(n)} \leqslant t} x_i^{(n)} - \sum_{t_i \leqslant t} x_i \Bigg| \leqslant \sum_{t_i \leqslant t} \delta \leqslant k\delta.$$

Therefore

$$\lim_{n \to \infty} |\psi^{(u)}(\eta_n)(t) - \psi^{(u)}(\eta)(t)| \leqslant k\delta,$$

and if we let $\delta \to 0$, it follows that $\psi^{(u)}(\eta_n)(t) \to \psi^{(u)}(\eta)(t)$ as $n \to \infty$. Put

$$v_i = \tau_i + r, \qquad i \in \{1, \ldots, p\}.$$

For any $\delta < u \wedge r$, relation (2.2) and the fact that $\eta \in \Lambda$ imply that the functions $\psi^{(u)}(\eta)$ and $\psi^{(u)}(\eta_n)$ $(n \geqslant n_\delta)$ are monotone on each of the intervals $[0, v_1], [v_1, v_2], \ldots, [v_p, 1]$. Now a combination of Proposition 2.4 and Lemma 12.9.2 in Whitt [69] yield that $d_{M_1}(\psi^{(u)}(\eta_n), \psi^{(u)}(\eta)) \to 0$ as $n \to \infty$. The application of Lemma 12.9.2 is justified by continuity of $\psi^{(u)}(\eta)$ in the boundary points $v_1, \ldots, v_p$. We conclude that $\psi^{(u)}$ is continuous at η. $\square$

Example 2.9. This example shows that we can not replace the set Λ in Lemma 2.8 by the set $M_p([0,1] \times \overline{\mathbb{E}}_u)$. Fix $u > 0$ and let

$$\eta_n = \delta_{(\frac{1}{2}, 2u)} + \delta_{(\frac{1}{2}+\frac{1}{n}, -2u)}, \quad n \geqslant 3.$$

Then $\eta_n \xrightarrow{v} \eta$ as $n \to \infty$, where

$$\eta = \delta_{(\frac{1}{2}, 2u)} + \delta_{(\frac{1}{2}, -2u)}.$$

Clearly $\eta \in M_p([0,1] \times \overline{\mathbb{E}}_u)$ (but $\eta \notin \Lambda$). It is straightforward to obtain $\psi^{(u)}(\eta_n)(t) = 2u \cdot 1_{[\frac{1}{2}, \frac{1}{2}+\frac{1}{n})}(t)$ and $\psi^{(u)}(\eta)(t) = 0$ for any $t \in [0,1]$. For all parametric representations $(r_n, v_n) \in \Pi(\psi^{(u)}(\eta_n))$ and $(r, v) \in \Pi(\psi^{(u)}(\eta))$ we have

$$\|v_n - v\|_{[0,1]} = 2u.$$

Therefore $d_{M_1}(\psi^{(u)}(\eta_n), \psi^{(u)}(\eta)) \geqslant 2u$ for all $n \geqslant 3$, yielding that $\psi^{(u)}(\eta_n)$ does not converge to $\psi^{(u)}(\eta)$ as $n \to \infty$. Hence $\psi^{(u)}$ is not continuous at η, and we conclude that $\psi^{(u)}$ is not continuous on the set $[0,1] \times \overline{\mathbb{E}}_u$.

The following lemma claims that under a certain assumption on the tail process, the point process $N^{(u)}$ defined in Theorem 1.36 almost surely belongs to the set Λ.

Lemma 2.10. *Assume that with probability one, the tail process $(Y_i)_{i \in \mathbb{Z}}$ in (1.12) has no two values of the opposite sign. Then $\mathrm{P}(N^{(u)} \in \Lambda) = 1$.*

Proof. From the definition of the tail process $(Y_i)_{i\in\mathbb{Z}}$ and Definition 1.5, we have for all $j \in \mathbb{Z}$ and $r \in (0,1)$,

$$\begin{aligned} \mathrm{P}(Y_j \in (1-r,1+r)) &= \lim_{n\to\infty} \frac{n\mathrm{P}\big(X_j/a_n \in (1-r,1+r),\, |X_0| > a_n\big)}{n\mathrm{P}(|X_0| > a_n)} \\ &\leqslant \lim_{n\to\infty} \frac{n\mathrm{P}\big(X_j/a_n \in (1-r,1+r)\big)}{n\mathrm{P}(|X_0| > a_n)} \\ &= \mu((1-r,1+r)). \end{aligned}$$

Now letting $r \to 0$ and taking into account the form of the limiting measure μ we obtain that

$$\mathrm{P}(Y_j = 1) \leqslant \mu(\{1\}) = 0,$$

i.e. $\mathrm{P}(Y_j = 1) = 0$. Similarly it holds that $\mathrm{P}(Y_j = -1) = 0$. Therefore $\mathrm{P}(Y_j = \pm 1) = 0$ for every j, and this implies

$$\begin{aligned} \mathrm{P}\Big(\sum_j \delta_{Y_j}(\{\pm 1\}) = 0\Big) &= \mathrm{P}\Big(\bigcap_j \{Y_j \neq \pm 1\}\Big) = 1 - \mathrm{P}\Big(\bigcup_j \{Y_j = \pm 1\}\Big) \\ &\geqslant 1 - \sum_j \mathrm{P}(Y_j = \pm 1) = 1, \end{aligned}$$

i.e. $\mathrm{P}\big(\sum_j \delta_{Y_j}(\{\pm 1\}) = 0\big) = 1$. From the definition of the processes $\sum_j \delta_{Z_{ij}}$ in Theorem 1.36, it follows that $\mathrm{P}\big(\sum_j \delta_{Z_{ij}}(\{\pm 1\}) = 0\big) = 1$ for every i. From here we immediately conclude that $\mathrm{P}(N^{(u)}([0,1] \times \{\pm u\}) = 0) = 1$. From the definition of the tail process $(Y_i)_{i\in\mathbb{Z}}$ we know that $\mathrm{P}(Y_i = \pm\infty) = 0$ for any $i \in \mathbb{Z}$. Therefore we obtain that $\mathrm{P}(N^{(u)}([0,1] \times \{\pm\infty\}) = 0) = 1$. Together with the fact that $\mathrm{P}(\sum_i \delta_{T_i^{(u)}}(\{0,1\}) = 0) = 1$, this implies $\mathrm{P}(N^{(u)} \in \Lambda_1) = 1$.

Further, the assumption that with probability one the tail process $(Y_i)_{i\in\mathbb{Z}}$ has no two values of the opposite sign yields $\mathrm{P}(N^{(u)} \in \Lambda_2) = 1$. □

Remark 2.11. It is straightforward to see that the conclusion of Lemma 2.10 holds if we replace the point process $N^{(u)}$ by $N^{(u)}\big|_{[0,1]\times\mathbb{E}_u}$.

Remark 2.12. Taking in account Lemma 2.8 and Lemma 2.10, we see that the summation functional $\psi^{(u)}$ is almost surely continuous with respect to the distribution of $N^{(u)}$.

The following proposition gives some sufficient conditions under which a Poisson process with state space $[0,1] \times \overline{\mathbb{E}}_u$ almost surely belongs to the set Λ. The proof of this result is a slight modification of the considerations in Resnick [60, p. 222].

Proposition 2.13. *Suppose N is a Poisson process with mean measure $\mathbb{LEB} \times \kappa$, where $\mathbb{LEB}$ is the Lebesgue measure on $[0,1]$ and κ is a Radon measure on $\overline{\mathbb{E}}_u$ such that $\kappa(\{\pm u, \pm\infty\}) = 0$. Then $\mathrm{P}(N \in \Lambda) = 1$.*

Proof. First, we have that

$$\mathbb{LEB} \times \kappa(\{0\} \times \mathbb{E}_u) = \mathbb{LEB}(\{0\}) \cdot \kappa(\mathbb{E}_u) = 0,$$

since $\kappa(\mathbb{E}_u) \leqslant \kappa(\overline{\mathbb{E}}_u) < \infty$. Taking into account the definition of the Poisson process (see Example 1.20), this implies that

$$\mathrm{P}(N(\{0\} \times \mathbb{E}_u) = 0) = 1.$$

Similarly, we have $\mathrm{P}(N(\{1\} \times \mathbb{E}_u) = 0) = 1$. Further, since

$$\mathbb{LEB} \times \kappa([0,1] \times \{\pm u, \pm\infty\}) = \mathbb{LEB}([0,1]) \cdot \kappa(\{\pm u, \pm\infty\}) = 0,$$

it follows that $\mathrm{P}(N([0,1] \times \{\pm u, \pm\infty\}) = 0) = 1$. Hence $\mathrm{P}(N \in \Lambda_1) = 1$.

One can write N in the form

$$N \stackrel{d}{=} \sum_{i=1}^{\xi} \delta_{(T_i,\, J_i)},$$

where ξ is a Poisson random variable with parameter $\mathbb{LEB} \times \kappa([0,1] \times \overline{\mathbb{E}}_u)$, $\{T_i,\, i \geqslant 1\}$ are i.i.d. uniformly distributed on $(0,1)$, $\{J_i,\, i \geqslant 1\}$ are i.i.d. with distribution

$\kappa(\overline{\mathbb{E}}_u \cap \cdot)/\kappa(\overline{\mathbb{E}}_u)$, and ξ is independent of $\{(T_i,\ J_i),\ i \geqslant 1\}$ (see Resnick [60, p. 143, 147]). Then

$$\begin{aligned} \mathrm{P}(\text{some vertical line contains two points of } N) &= \mathrm{P}\Big(\bigcup_{1 \leqslant i < j \leqslant \xi} \{T_i = T_j\} \Big) \\ &\leqslant \sum_{1 \leqslant i < j < \infty} \mathrm{P}(T_i = T_j) = 0 \end{aligned}$$

This suffices to conclude that $\mathrm{P}(N \in \Lambda_2) = 1$. $\square$

2.3 Main theorem

Let (X_n) be a strictly stationary sequence of random variables, jointly regularly varying with index $\alpha \in (0,2)$ and tail process $(Y_i)_{i \in \mathbb{Z}}$. The theorem below gives conditions under which its partial sum process satisfies a nonstandard functional limit theorem with a non-Gaussian α–stable Lévy process as a limit. Recall that the distribution of a Lévy process $V(\,\cdot\,)$ is characterized by its characteristic triple, i.e. the characteristic triple (a, ν, b) of the infinitely divisible distribution of $V(1)$. The description of the characteristic triple of the limit process will be in terms of the measures $\nu^{(u)}$ $(u > 0)$ on $\mathbb{E}$ defined for $x > 0$ by

$$\begin{aligned} \nu^{(u)}(x, \infty] &= u^{-\alpha}\, \mathrm{P}\bigg(u \sum_{i \geqslant 0} Y_i\, 1_{\{|Y_i| > 1\}} > x,\ \sup_{i \leqslant -1} |Y_i| \leqslant 1 \bigg), \\ \nu^{(u)}[-\infty, -x) &= u^{-\alpha}\, \mathrm{P}\bigg(u \sum_{i \geqslant 0} Y_i\, 1_{\{|Y_i| > 1\}} < -x,\ \sup_{i \leqslant -1} |Y_i| \leqslant 1 \bigg). \end{aligned} \tag{2.3}$$

In the case $\alpha \in [1,2)$, we will need to assume that the contribution of the smaller increments of the partial sum process is close to its expectation.

Condition 2.14. For all $\delta > 0$,

$$\lim_{u \downarrow 0} \limsup_{n \to \infty} \mathrm{P}\bigg[\max_{1 \leqslant k \leqslant n} \bigg| \sum_{i=1}^{k} \bigg(\frac{X_i}{a_n} 1_{\big\{ \frac{|X_i|}{a_n} \leqslant u \big\}} - \mathrm{E}\bigg(\frac{X_i}{a_n} 1_{\big\{ \frac{|X_i|}{a_n} \leqslant u \big\}} \bigg) \bigg) \bigg| > \delta \bigg] = 0.$$

Condition 2.14 holds for instance if $(X_n)_n$ is ρ–mixing at a certain rate; see Proposition 2.19 in Section 2.4, where some variations of the following theorem are discussed as well.

Theorem 2.15. *Let $(X_n)_{n\in\mathbb{N}}$ be a strictly stationary sequence of random variables, regularly varying with index $\alpha \in (0,2)$, and such that its tail process $(Y_i)_{i\in\mathbb{Z}}$ almost surely has no two values of the opposite sign. Suppose that the mixing condition $\mathcal{A}'(a_n)$ and the anti-clustering condition $\mathcal{AC}(a_n)$ hold, where (a_n) is a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. If $\alpha \in [1,2)$, also suppose that Condition 2.14 holds. Then the partial sum stochastic process*

$$V_n(t) = \sum_{k=1}^{\lfloor nt \rfloor} \frac{X_k}{a_n} - \lfloor nt \rfloor \mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\big\{ \frac{|X_1|}{a_n} \leqslant 1 \big\}} \bigg), \qquad t \in [0,1], \tag{2.4}$$

satisfies

$$V_n \xrightarrow{d} V, \qquad n \to \infty,$$

in $D[0,1]$ endowed with the M_1 topology, where $V(\,\cdot\,)$ is an α–stable Lévy process with characteristic triple $(0,\nu,b)$, where

$$b = \lim_{u \to 0} \bigg[\int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\nu^{(u)}(dx) - \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx) \bigg]$$

and ν is the vague limit of $\nu^{(u)}$ as $u \downarrow 0$, with $\nu^{(u)}$ as in (2.3) and μ as in (1.8).

Proof. Note that from Theorem 1.36 and the fact that $|Y_n| \to 0$ almost surely as $|n| \to \infty$, the random variables

$$u \sum_j Z_{ij} 1_{\{|Z_{ij}|>1\}}$$

are i.i.d. and almost surely finite. Define

$$\widehat{N}^{(u)} = \sum_i \delta_{(T_i^{(u)},\, u \sum_j Z_{ij} 1_{\{|Z_{ij}|>1\}})}.$$

Then by Proposition 5.3 in Resnick [60], $\widehat{N}^{(u)}$ is a Poisson process with mean measure

$$\theta u^{-\alpha} \mathbb{LEB} \times F^{(u)}, \tag{2.5}$$

where $F^{(u)}$ is the distribution of the random variable $u \sum_j Z_{1j} 1_{\{|Z_{1j}|>1\}}$. But for $0 \leqslant s < t \leqslant 1$ and $x > 0$, using the fact that the distribution of $\sum_j \delta_{Z_{1j}}$ is equal to the one of $\sum_j \delta_{Y_j}$ conditionally on the event $\{\sup_{i \leqslant -1} |Y_i| \leqslant 1\}$ and the fact that $\mathrm{P}(\sup_{i \leqslant -1} |Y_i| \leqslant 1) = \theta > 0$ (see Section 1.6), we have

$$\begin{aligned}
&\theta u^{-\alpha} \mathbb{LEB} \times F^{(u)}([s,t] \times (x,\infty]) = \theta u^{-\alpha}(t-s) F^{(u)}((x,\infty]) \\
&= \theta u^{-\alpha}(t-s) \mathrm{P}\bigg(u \sum_j Z_{1j} 1_{\{|Z_{1j}|>1\}} > x \bigg) \\
&= \theta u^{-\alpha}(t-s) \mathrm{P}\bigg(u \sum_j Y_j 1_{\{|Y_j|>1\}} > x \,\bigg|\, \sup_{i \leqslant -1} |Y_i| \leqslant 1 \bigg) \\
&= \theta u^{-\alpha}(t-s) \frac{\mathrm{P}\big(u \sum_j Y_j 1_{\{|Y_j|>1\}} > x,\, \sup_{i \leqslant -1} |Y_i| \leqslant 1\big)}{\mathrm{P}(\sup_{i \leqslant -1} |Y_i| \leqslant 1)} \\
&= u^{-\alpha}(t-s) \mathrm{P}\bigg(u \sum_j Y_j 1_{\{|Y_j|>1\}} > x,\, \sup_{i \leqslant -1} |Y_i| \leqslant 1 \bigg) \\
&= \mathbb{LEB} \times \nu^{(u)}([s,t] \times (x,\infty]).
\end{aligned}$$

The same can be done for the sets of the form $[s,t] \times [-\infty, -x)$, so that the mean measure in (2.5) is equal to $\mathbb{LEB} \times \nu^{(u)}$.

Consider now

$$\psi^{(u)}(N_n \,|\, {}_{[0,1] \times \mathbb{E}_u})(\,\cdot\,) = \sum_{i/n \leq \,\cdot} \frac{X_i}{a_n} 1_{\left\{ \frac{|X_i|}{a_n} > u \right\}},$$

which by Corollary 1.38, Lemma 2.8, Lemma 2.10 (in fact Remark 2.11) and the continuous mapping theorem (see for instance Theorem 3.1 in Resnick [60]) converges in distribution in $D[0,1]$ under the M_1 metric to

$$\psi^{(u)}(N^{(u)}\big|_{[0,1] \times \mathbb{E}_u})(\,\cdot\,) = \sum_{T_i^{(u)} \leqslant \,\cdot} \sum_j u Z_{ij} 1_{\{|Z_{ij}|>1\}}.$$

Let

$$\widetilde{N}^{(u)} = \sum_i \delta_{(T_i,\, K_i^{(u)})}$$

be a Poisson process with mean measure $\mathbb{LEB} \times \nu^{(u)}$. Since

$$\psi^{(u)}(N^{(u)}\big|_{[0,1]\times\mathbb{E}_u}) = \psi^{(u)}(\widehat{N}^{(u)}) \stackrel{d}{=} \psi^{(u)}(\widetilde{N}^{(u)}),$$

we obtain

$$L_n^{(u)}(\,\cdot\,) := \sum_{i=1}^{\lfloor n\,\cdot\,\rfloor} \frac{X_i}{a_n} 1_{\left\{\frac{|X_i|}{a_n}>u\right\}} \xrightarrow{d} L^{(u)}(\,\cdot\,) := \sum_{T_i \leqslant\,\cdot} K_i^{(u)}, \quad \text{as } n\to\infty, \tag{2.6}$$

in $D[0,1]$ under the M_1 metric. From relation (1.4) (in the one-dimensional case), Proposition 1.2 and Theorem 1.6 (iii) we have, for any $t \in [0,1]$, as $n \to \infty$,

$$\begin{aligned} \lfloor nt \rfloor \mathrm{E}\left(\frac{X_1}{a_n} 1_{\left\{u<\frac{|X_1|}{a_n}\leqslant 1\right\}} \right) &= \frac{\lfloor nt \rfloor}{n} \int_{\{x\,:\,u<|x|\leqslant 1\}} x n \mathrm{P}\left(\frac{X_1}{a_n} \in dx \right) \\ &\to t \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx). \end{aligned} \tag{2.7}$$

This convergence is uniform in t and hence

$$\lfloor n\,\cdot\,\rfloor \mathrm{E}\left(\frac{X_1}{a_n} 1_{\left\{u<\frac{|X_1|}{a_n}\leqslant 1\right\}} \right) \to (\,\cdot\,) \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx) \tag{2.8}$$

in the M_1 metric on $D[0,1]$.

Put

$$a_u = \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx),$$

and define the function $x^{(u)} \colon [0,1] \to \mathbb{R}$ by $x^{(u)}(t) = t a_u$. The function $x^{(u)}$ is continuous, and hence it belongs to $D[0,1]$. Define now $h \colon D[0,1] \to D[0,1]$ by $h(x) = x - x^{(u)}$. An application of Proposition 2.5 yields that h is a continuous function. Hence by the continuous mapping theorem we obtain $h(L_n^{(u)}) \xrightarrow{d} h(L^{(u)})$, i.e.

$$L_n^{(u)} - x^{(u)} \xrightarrow{d} L^{(u)} - x^{(u)}, \quad \text{as } n \to \infty, \tag{2.9}$$

in $D[0,1]$ under the M_1 metric. Define

$$V_n^{(u)}(\,\cdot\,) := \sum_{i=1}^{\lfloor n\,\cdot\,\rfloor} \frac{X_i}{a_n} 1_{\left\{\frac{|X_i|}{a_n} > u\right\}} - \lfloor n\,\cdot\,\rfloor \mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\left\{u < \frac{|X_1|}{a_n} \leqslant 1\right\}}\bigg),$$

$$V^{(u)}(\,\cdot\,) := L^{(u)}(\,\cdot\,) - x^{(u)}(\,\cdot\,) = \sum_{T_i \leqslant\,\cdot} K_i^{(u)} - (\,\cdot\,) \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx).$$

Next, we show that, for any $\delta > 0$,

$$\lim_{n\to\infty} \mathrm{P}[d_{M_1}(L_n^{(u)} - x^{(u)}, V_n^{(u)}) > \delta] = 0. \tag{2.10}$$

Since Skorohod's M_1 metric on $D[0,1]$ is bounded above by the uniform metric on $D[0,1]$ (see the second statement in Proposition 2.2 and relation (2.1)) and $|\lfloor x\rfloor - x| \leqslant 1$ for every $x \in \mathbb{R}$, we have

$$\begin{aligned}
&\mathrm{P}[d_{M_1}(L_n^{(u)} - x^{(u)}, V_n^{(u)}) > \delta] \\
&\quad \leqslant \mathrm{P}\bigg[\sup_{0\leqslant t\leqslant 1} \bigg|\lfloor nt\rfloor \mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\left\{u<\frac{|X_1|}{a_n}\leqslant 1\right\}}\bigg) - t a_u\bigg| > \delta\bigg] \\
&\quad \leqslant \mathrm{P}\bigg[\sup_{0\leqslant t\leqslant 1} |\lfloor nt\rfloor - nt| \cdot \bigg|\mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\left\{u<\frac{|X_1|}{a_n}\leqslant 1\right\}}\bigg)\bigg| > \frac{\delta}{2}\bigg] \\
&\qquad + \mathrm{P}\bigg[\sup_{0\leqslant t\leqslant 1} |t| \cdot \bigg|n\mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\left\{u<\frac{|X_1|}{a_n}\leqslant 1\right\}}\bigg) - a_u\bigg| > \frac{\delta}{2}\bigg] \\
&\quad \leqslant \mathrm{P}\bigg[\bigg|\mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\left\{u<\frac{|X_1|}{a_n}\leqslant 1\right\}}\bigg)\bigg| > \frac{\delta}{2}\bigg] + \mathrm{P}\bigg[\bigg|n\mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\left\{u<\frac{|X_1|}{a_n}\leqslant 1\right\}}\bigg) - a_u\bigg| > \frac{\delta}{2}\bigg].
\end{aligned}$$

From this, using relation (2.7) (with $t = 1$), we obtain

$$\limsup_{n\to\infty} \mathrm{P}[d_{M_1}(L_n^{(u)} - x^{(u)}, V_n^{(u)}) > \delta] = 0.$$

Therefore (2.10) holds. Now from relations (2.9), (2.10) and Slutsky's theorem (see for instance Theorem 3.4 in Resnick [60]), we obtain

$$V_n^{(u)}(\,\cdot\,) \xrightarrow{d} V^{(u)}(\,\cdot\,), \qquad \text{as } n \to \infty, \tag{2.11}$$

in $D[0,1]$ under the M_1 metric. The limit in (2.11) can be rewritten as

$$\sum_{T_i \leq \cdot} K_i^{(u)} - (\,\cdot\,) \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\nu^{(u)}(dx) \\ + (\,\cdot\,)\Big(\int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\nu^{(u)}(dx) - \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx)\Big).$$

Note that the first two terms represent a Lévy–Ito representation of the Lévy process with characteristic triple $(0,\nu^{(u)},0)$, see Resnick [60, p. 150]. The remaining term is just a linear function of the form $t \mapsto t\,b_u$. As a consequence, the process $V^{(u)}$ is a Lévy process for each $u<1$, with characteristic triple $(0,\nu^{(u)},b_u)$, where

$$b_u = \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\nu^{(u)}(dx) - \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx).$$

The next step is to show that $V^{(u)}(1)$ converges to an α–stable random variable as $u \to 0$. Here we shall use some facts from the proof of Theorem 3.1 in Davis and Hsing [24]. First we have to show that all conditions from this theorem hold. Since the random process (X_n) is regularly varying with index $\alpha \in (0,2)$, and conditions $\mathcal{A}'(a_n)$ and $\mathcal{AC}(a_n)$ hold, from Theorem 2.7 in Davis and Hsing [24] it follows that the point process N_n^*, as defined in Section 1.6, converges to some N^*. From Proposition 4.2 in Basrak and Segers [10] it follows that the case $N^*=o$ can never occur. Hence relation (3.1) in Davis and Hsing [24] holds. Condition (3.2) in [24] holds, since it is implied by Condition 2.14.

Theorem 12.5.1 (iv) in Whitt [69] implies that the function $\pi\colon D[0,1] \to \mathbb{R}$ defined by $\pi(x)=x(1)$ is continuous. Hence, by (2.11) and the continuous mapping theorem, as $n \to \infty$,

$$V_n^{(u)}(1) \xrightarrow{d} V^{(u)}(1). \tag{2.12}$$

Now we distinguish two cases:

Case 1. $\alpha \in (0,1)$. From relation (3.4) in [24] we have, as $n \to \infty$,

$$V_n^{(u)}(1) + n\mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\big\{u < \frac{|X_1|}{a_n} \leqslant 1\big\}}\bigg) \xrightarrow{d} T_u(N^*),$$

where T_u is the mapping from $M_p(\mathbb{E})$ into $\mathbb{R}$ defined by

$$T_u\Big(\sum_{i=1}^{\infty} \delta_{x_i}\Big) = \sum_{i=1}^{\infty} x_i 1_{\{u < |x_i| < \infty\}}.$$

This together with (2.7) (with $t = 1$), by Corollary 2 in Chow and Teicher [20, p. 272], imply

$$V_n^{(u)}(1) \xrightarrow{d} T_u(N^*) - \int_{\{x \,:\, u < |x| \leqslant 1\}} x \, \mu(dx).$$

Hence, from (2.12) we see that

$$V^{(u)}(1) \stackrel{d}{=} T_u(N^*) - \int_{\{x \,:\, u < |x| \leqslant 1\}} x \, \mu(dx).$$

By relation (3.5) in [24], $T_u(N^*) \xrightarrow{d} T_0(N^*)$ as $u \to 0$, and the limit is an α–stable random variable. From the representation of the measure μ in (1.8) we obtain

$$\lim_{u \to 0} \int_{\{x \,:\, u < |x| \leqslant 1\}} x \, \mu(dx) = (p-q)\frac{\alpha}{1-\alpha} \cdot \lim_{u \to 0}(1 - u^{1-\alpha}) = (p-q)\frac{\alpha}{1-\alpha}.$$

Now a new application of Corollary 2 in [20] implies, as $u \to 0$,

$$V^{(u)}(1) \stackrel{d}{=} T_u(N^*) - \int_{\{x \,:\, u < |x| \leqslant 1\}} x \, \mu(dx) \xrightarrow{d} T_0(N^*) - (p-q)\frac{\alpha}{1-\alpha}.$$

Since $T_0(N^*)$ is α–stable, the random variable $T_0(N^*) - (p-q)\frac{\alpha}{1-\alpha}$ is also α–stable (this fact is a consequence of the representation of a stable random variable given in Theorem 1.45). Thus $V^{(u)}(1)$ converges to an α–stable random variable.

Case 2. $\alpha \in [1,2)$. From relation (3.8) in [24], we have, as $n \to \infty$,

$$V_n^{(u)}(1) \xrightarrow{d} T_u(N^*) - \int_{\{x \,:\, u < |x| \leqslant 1\}} x \, \mu(dx).$$

Hence, from (2.12) we see that

$$V^{(u)}(1) \stackrel{d}{=} T_u(N^*) - \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx).$$

By relation (3.9) in [24],

$$T_u(N^*) - \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx) \xrightarrow{d} \text{some } S, \qquad \text{as } u \to 0,$$

where the limit is an α–stable random variable. Thus $V^{(u)}(1) \xrightarrow{d} S$.

In both cases we conclude that $V^{(u)}(1)$ converges, as $u \to 0$, to an α–stable random variable. Since every stable random variable is infinitely divisible, by Theorem 13.12 in Kallenberg [40], there exists a Lévy process $V(\,\cdot\,)$ such that

$$V^{(u)}(1) \xrightarrow{d} V(1).$$

Hence by Theorem 13.17 in [40], there exist some processes $\widetilde{V}^{(u)} \stackrel{d}{=} V^{(u)}$ with

$$\lim_{u\to 0} \mathrm{P}\biggl(\sup_{0\leqslant t\leqslant 1} |\widetilde{V}^{(u)}(t) - V(t)| > \delta \biggr) = 0,$$

for every $\delta > 0$. Since the M_1 metric on $D[0,1]$ is bounded above by the uniform metric on $D[0,1]$, it follows that

$$\lim_{u\to 0} \mathrm{P}(d_{M_1}(\widetilde{V}^{(u)}, V) > \delta) = 0,$$

and this immediately implies $\widetilde{V}^{(u)}(\,\cdot\,) \xrightarrow{d} V(\,\cdot\,)$ (see for instance Theorem 3.4 in Resnick [60]), i.e.

$$V^{(u)}(\,\cdot\,) \xrightarrow{d} V(\,\cdot\,), \quad \text{as } u \to 0, \tag{2.13}$$

in $D[0,1]$ with the M_1 metric. The process $V(\,\cdot\,)$ has characteristic triple $(0,\nu,b)$, where ν is the vague limit of $\nu^{(u)}$ as $u \to 0$ and $b = \lim_{u\to 0} b_u$, see Theorem 13.14 in Kallenberg [40]. Since the random variable $V(1)$ has an α–stable distribution, it follows that the process $V(\,\cdot\,)$ is α–stable.

If we show that

$$\lim_{u\downarrow 0}\limsup_{n\to\infty} \mathrm{P}[d_{M_1}(V_n^{(u)}, V_n) > \delta] = 0$$

for any $\delta > 0$, then from (2.11), (2.13) and Theorem 3.5 in Resnick [60] we will have, as $n \to \infty$,

$$V_n(\,\cdot\,) \xrightarrow{d} V(\,\cdot\,)$$

in $D[0,1]$ with the M_1 metric. Once again, since the M_1 metric on $D[0,1]$ is bounded above by the uniform metric on $D[0,1]$, it suffices to show that

$$\lim_{u\downarrow 0}\limsup_{n\to\infty} \mathrm{P}\biggl(\sup_{0\leqslant t\leqslant 1} |V_n^{(u)}(t) - V_n(t)| > \delta\biggr) = 0. \tag{2.14}$$

Recalling the definitions, we have

$$\begin{aligned}
&\lim_{u\downarrow 0}\limsup_{n\to\infty} \mathrm{P}\biggl(\sup_{0\leqslant t\leqslant 1} |V_n^{(u)}(t) - V_n(t)| > \delta\biggr)\\
&\quad= \lim_{u\downarrow 0}\limsup_{n\to\infty} \mathrm{P}\biggl[\sup_{0\leqslant t\leqslant 1}\biggl|\sum_{i=1}^{\lfloor nt\rfloor}\frac{X_i}{a_n}1_{\bigl\{\frac{|X_i|}{a_n}\leqslant u\bigr\}} - \lfloor nt\rfloor \mathrm{E}\biggl(\frac{X_1}{a_n}1_{\bigl\{\frac{|X_1|}{a_n}\leqslant u\bigr\}}\biggr)\biggr| > \delta\biggr]\\
&\quad= \lim_{u\downarrow 0}\limsup_{n\to\infty} \mathrm{P}\biggl[\sup_{0\leqslant t\leqslant 1}\biggl|\sum_{i=1}^{\lfloor nt\rfloor}\biggl\{\frac{X_i}{a_n}1_{\bigl\{\frac{|X_i|}{a_n}\leqslant u\bigr\}} - \mathrm{E}\biggl(\frac{X_i}{a_n}1_{\bigl\{\frac{|X_i|}{a_n}\leqslant u\bigr\}}\biggr)\biggr\}\biggr| > \delta\biggr]\\
&\quad= \lim_{u\downarrow 0}\limsup_{n\to\infty} \mathrm{P}\biggl[\max_{1\leqslant k\leqslant n}\biggl|\sum_{i=1}^{k}\biggl\{\frac{X_i}{a_n}1_{\bigl\{\frac{|X_i|}{a_n}\leqslant u\bigr\}} - \mathrm{E}\biggl(\frac{X_i}{a_n}1_{\bigl\{\frac{|X_i|}{a_n}\leqslant u\bigr\}}\biggr)\biggr\}\biggr| > \delta\biggr].
\end{aligned}$$

Therefore we have to show

$$\lim_{u\downarrow 0}\limsup_{n\to\infty} \mathrm{P}\biggl[\max_{1\leqslant k\leqslant n}\biggl|\sum_{i=1}^{k}\biggl\{\frac{X_i}{a_n}1_{\bigl\{\frac{|X_i|}{a_n}\leqslant u\bigr\}} - \mathrm{E}\biggl(\frac{X_i}{a_n}1_{\bigl\{\frac{|X_i|}{a_n}\leqslant u\bigr\}}\biggr)\biggr\}\biggr| > \delta\biggr] = 0. \tag{2.15}$$

For $\alpha \in [1,2)$ this relation is simply Condition 2.14. Therefore it remains to show (2.15) for the case when $\alpha \in (0,1)$. Hence assume $\alpha \in (0,1)$. For an arbitrary (and fixed) $\delta > 0$ define

$$I(u,n) = \mathrm{P}\biggl[\max_{1\leqslant k\leqslant n}\biggl|\sum_{i=1}^{k}\biggl\{\frac{X_i}{a_n}\,1_{\bigl\{\frac{|X_i|}{a_n}\leqslant u\bigr\}} - \mathrm{E}\biggl(\frac{X_i}{a_n}\,1_{\bigl\{\frac{|X_i|}{a_n}\leqslant u\bigr\}}\biggr)\biggr\}\biggr| > \delta\biggr].$$

Using stationarity and Chebyshev's inequality we get the bound

$$\begin{aligned}
I(u,n) &\leqslant \mathrm{P}\Big[\sum_{i=1}^{n}\Big|\frac{X_i}{a_n}1_{\{\frac{|X_i|}{a_n}\leqslant u\}}-\mathrm{E}\Big(\frac{X_i}{a_n}1_{\{\frac{|X_i|}{a_n}\leqslant u\}}\Big)\Big|>\delta\Big]\\
&\leqslant \delta^{-1}\mathrm{E}\Big[\sum_{i=1}^{n}\Big|\frac{X_i}{a_n}1_{\{\frac{|X_i|}{a_n}\leqslant u\}}-\mathrm{E}\Big(\frac{X_i}{a_n}1_{\{\frac{|X_i|}{a_n}\leqslant u\}}\Big)\Big|\Big]\\
&\leqslant 2\delta^{-1}n\mathrm{E}\Big(\frac{|X_1|}{a_n}1_{\{\frac{|X_1|}{a_n}\leqslant u\}}\Big)\\
&= 2\delta^{-1}u\cdot n\mathrm{P}(|X_1|>a_n)\cdot\frac{\mathrm{P}(|X_1|>ua_n)}{\mathrm{P}(|X_1|>a_n)}\cdot\frac{\mathrm{E}(|X_1|1_{\{|X_1|\leqslant ua_n\}})}{ua_n\mathrm{P}(|X_1|>ua_n)}. \qquad (2.16)
\end{aligned}$$

Since X_1 is a regularly varying random variable with index α, an application of Proposition 1.8 gives

$$\frac{\mathrm{P}(|X_1|>ua_n)}{\mathrm{P}(|X_1|>a_n)}\to u^{-\alpha},$$

as $n\to\infty$. By Theorem 1.12

$$\lim_{n\to\infty}\frac{\mathrm{E}(|X_1|1_{\{|X_1|\leqslant ua_n\}})}{ua_n\mathrm{P}(|X_1|>ua_n)}=\frac{\alpha}{1-\alpha}.$$

Thus from (2.16), taking into account the fact that $n\mathrm{P}(|X_1|>a_n)\to 1$ as $n\to\infty$, we get

$$\limsup_{n\to\infty}I(u,n)\leq 2\delta^{-1}\frac{\alpha}{1-\alpha}u^{1-\alpha}.$$

Letting $u\to 0$, since $1-\alpha>0$, we finally obtain

$$\lim_{u\downarrow 0}\limsup_{n\to\infty}I(u,n)=0,$$

and relation (2.15) holds. Therefore $V_n\xrightarrow{d}V$ as $n\to\infty$ in $D[0,1]$ endowed with the M_1 topology. $\square$

2.4 Discussion

In this section we revisit the conditions and the conclusions of Theorem 2.15 and provide some additional insights. Since the measure ν is the Lévy measure of a stable random variable $V(1)$, it can be represented in the form given in Remark 1.47 (see Remark 2.16 below). In case $\alpha \in (0,1)$, the centering function in the definition of V_n can be removed (see Remark 2.17 below). In the other case, $\alpha \in [1,2)$, the centering function cannot be omitted, and one way of checking Condition 2.14 is via ρ–mixing as we show in Proposition 2.19. Finally, in Theorem 2.15 we can not replace the M_1 topology by the J_1 topology (see Remark 2.20 below).

Remark 2.16. The Lévy measure ν satisfies the scaling property

$$\nu(sB) = s^{-\alpha}\nu(B), \qquad s > 0,\ B \in \mathcal{B}(\mathbb{E}),$$

(see Theorem 1.44). In particular, as in Remark 1.47, ν can be written as

$$\nu(dx) = \big(c_1\, 1_{(0,\infty)}(x) + c_2\, 1_{(-\infty,0)}(x)\big)\, |x|^{-\alpha-1}\, dx,$$

for some nonnegative constants c_1 and c_2, and therefore $\nu(\{x\}) = 0$ for every $x \in \mathbb{E}$. Thus, from Theorem 1.4 and the fact that the spectral process $(\Theta_i)_{i\in\mathbb{Z}}$ is independent of $|Y_0|$ (see Theorem 3.1 in Basrak and Segers [10]), we have

$$\begin{aligned}
c_1 &= \alpha\,\nu(1,\infty] = \lim_{u\to 0} \alpha\, \nu^{(u)}(1,\infty] \\
&= \lim_{u\to 0} \alpha\, u^{-\alpha}\, \mathrm{P}\bigg(u \sum_{i\geqslant 0} Y_i\, 1_{\{|Y_i|>1\}} > 1,\ \sup_{i\leqslant -1} |Y_i| \leqslant 1 \bigg) \\
&= \lim_{u\to 0} \alpha\, u^{-\alpha} \int_1^\infty \mathrm{P}\bigg(u \sum_{i\geqslant 0} r\Theta_i\, 1_{\{r|\Theta_i|>1\}} > 1,\ \sup_{i\leqslant -1} r|\Theta_i| \leqslant 1 \bigg)\, d(-r^{-\alpha}) \\
&= \lim_{u\to 0} \alpha \int_u^\infty \mathrm{P}\bigg(\sum_{i\geqslant 0} r\Theta_j\, 1_{\{r|\Theta_j|>u\}} > 1,\ \sup_{i\leqslant -1} r|\Theta_i| \leqslant u \bigg)\, d(-r^{-\alpha}),
\end{aligned}$$

and similarly

$$c_2 = \lim_{u\to 0} \alpha \int_u^\infty \mathrm{P}\bigg(\sum_{i\geqslant 0} r\Theta_j\, 1_{\{r|\Theta_j|>u\}} < -1,\ \sup_{i\leqslant -1} r|\Theta_i| \leqslant u \bigg)\, d(-r^{-\alpha}).$$

Remark 2.17. If $\alpha \in (0,1)$, the centering function in the definition of the stochastic process $V_n(\,\cdot\,)$ can be removed and this removing affects the characteristic triple of the limiting process in the way we describe here.

First, note that for arbitrary random variable X and $x > 0$ it holds

$$X 1_{\{|X| \leqslant x\}} = X^+ 1_{\{X^+ \leqslant x\}} - X^- 1_{\{X^- \leqslant x\}},$$

where $X^+ = \max\{X, 0\}$ and $X^- = \max\{-X, 0\}$ are the positive and negative parts of X.

From relation (1.7) and the fact that the sequence (a_n) is chosen such that $n\mathrm{P}(|X_1| > a_n) \to 1$, we obtain, as $n \to \infty$,

$$n\mathrm{P}(X_1^+ > a_n) = \frac{\mathrm{P}(X_1 > a_n)}{\mathrm{P}(|X_1| > a_n)} \cdot n\mathrm{P}(|X_1| > a_n) \to p,$$

and similarly $n\mathrm{P}(X_1^- > a_n) \to q = 1 - p$. We distinguish two cases:

Case 1. $p \in (0,1)$. Since X_1 is regularly varying with index α and $p, q > 0$, X_1^+ and X_1^- are also regularly varying with index α (we can see this through the statement in Remark 1.10 equivalent to regular variation). Therefore by Theorem 1.12, as $n \to \infty$,

$$\begin{aligned}
n\,\mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\big\{\frac{|X_1|}{a_n} \leqslant 1\big\}}\bigg) &= n\,\mathrm{E}\bigg(\frac{X_1^+}{a_n} 1_{\big\{\frac{X_1^+}{a_n} \leqslant 1\big\}}\bigg) - n\,\mathrm{E}\bigg(\frac{X_1^-}{a_n} 1_{\big\{\frac{X_1^-}{a_n} \leqslant 1\big\}}\bigg) \\
&= n\mathrm{P}(X_1^+ > a_n) \cdot \frac{\mathrm{E}\big(X_1^+ 1_{\{X_1^+ \leqslant a_n\}}\big)}{a_n \mathrm{P}(X_1^+ > a_n)} - n\mathrm{P}(X_1^- > a_n) \cdot \frac{\mathrm{E}\big(X_1^- 1_{\{X_1^- \leqslant a_n\}}\big)}{a_n \mathrm{P}(X_1^- > a_n)} \\
&\to (p - q)\frac{\alpha}{1-\alpha}.
\end{aligned}$$

Case 2. $p = 0$ or 1. Assume $p = 1$ (the case when $p = 0$ can be treated similarly and is here omitted). Then X_1^+ is regularly varying with index α and therefore it holds that

$$n\,\mathrm{E}\bigg(\frac{X_1^+}{a_n} 1_{\big\{\frac{X_1^+}{a_n} \leqslant 1\big\}}\bigg) \to p\,\frac{\alpha}{1-\alpha} = \frac{\alpha}{1-\alpha}, \qquad \text{as } n \to \infty.$$

This and Theorem 1.12 imply that, as $n \to \infty$,

$$\begin{aligned} n\,\mathrm{E}\bigg(\frac{X_1^-}{a_n}\,1_{\big\{\frac{X_1^-}{a_n}\leqslant 1\big\}}\bigg) &= n\,\mathrm{E}\bigg(\frac{|X_1|}{a_n}\,1_{\big\{\frac{|X_1|}{a_n}\leqslant 1\big\}}\bigg) - n\,\mathrm{E}\bigg(\frac{X_1^+}{a_n}\,1_{\big\{\frac{X_1^+}{a_n}\leqslant 1\big\}}\bigg) \\ &\to \frac{\alpha}{1-\alpha} - \frac{\alpha}{1-\alpha} = 0. \end{aligned}$$

Therefore, as $n \to \infty$,

$$\begin{aligned} n\,\mathrm{E}\bigg(\frac{X_1}{a_n}\,1_{\big\{\frac{|X_1|}{a_n}\leqslant 1\big\}}\bigg) &= n\,\mathrm{E}\bigg(\frac{X_1^+}{a_n}\,1_{\big\{\frac{X_1^+}{a_n}\leqslant 1\big\}}\bigg) - n\,\mathrm{E}\bigg(\frac{X_1^-}{a_n}\,1_{\big\{\frac{X_1^-}{a_n}\leqslant 1\big\}}\bigg) \\ &\to \frac{\alpha}{1-\alpha} - 0 = (p-q)\frac{\alpha}{1-\alpha}. \end{aligned}$$

In both cases we conclude that, as $n \to \infty$,

$$\lfloor n\,\cdot\,\rfloor \mathrm{E}\bigg(\frac{X_1}{a_n}\,1_{\big\{\frac{|X_1|}{a_n}\leqslant 1\big\}}\bigg) \to (\,\cdot\,)(p-q)\frac{\alpha}{1-\alpha}$$

in the M_1 metric on $D[0,1]$, which leads to

$$\sum_{k=1}^{\lfloor n\,\cdot\,\rfloor} \frac{X_k}{a_n} \xrightarrow{d} V(\,\cdot\,) + (\,\cdot\,)(p-q)\frac{\alpha}{1-\alpha}$$

in $D[0,1]$ endowed with the M_1 topology. The characteristic triple of the limiting process is therefore $(0, \nu, b')$ with $b' = b + (p-q)\alpha/(1-\alpha)$.

Example 2.18. Consider the process

$$X_n = Z_n - Z_{n-1}, \qquad n \in \mathbb{Z},$$

where (Z_n) is an i.i.d. sequence of random variables with common distribution given by the probability density function

$$f(x) = \begin{cases} (\alpha/2)|x|^{-(\alpha+1)} & \text{if } |x| \geqslant 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$. Then the process (X_n) is regularly varying with index α and

$$\sum_{k=1}^{\lfloor n\cdot\rfloor} \frac{X_k}{a_n} = \frac{Z_{\lfloor n\cdot\rfloor} - Z_0}{a_n} \xrightarrow{\text{fidi}} 0. \tag{2.17}$$

But since, as is known, $\sup_{t\in[0,1]} Z_{\lfloor nt\rfloor}/a_n$ converges in distribution to a nonzero limit (see Proposition 7.2 in Resnick [60]) and the functional $\sup_{t\in[0,1]}$ is continuous in the M_1 topology (see Skorohod [65]), the "fidi" convergence in (2.17) can not be replaced by convergence in distribution in the M_1 topology. Therefore the process $V_n(\,\cdot\,)$ does not converge in distribution in $D[0,1]$ endowed with the M_1 topology.

The process (X_n) belongs to a class of finite order MA processes. In Section 4.1 we will analyze these processes in detail, but let say here that the only condition in Theorem 2.15 that (X_n) does not satisfy is the one on the tail process. Indeed, if (Y_n) is the tail process of (X_n), then a standard regular variation argument and Lemma 1.2. in Cline [21] imply

$$
\begin{aligned}
\mathrm{P}(Y_0>1,Y_1<-1) &= \lim_{x\to\infty}\frac{\mathrm{P}(X_0>x,X_1<-x)}{\mathrm{P}(|X_0|>x)} \\
&= \lim_{x\to\infty}\frac{\mathrm{P}(Z_0-Z_{-1}>x,Z_1-Z_0<-x)}{\mathrm{P}(|X_0|>x)} \\
&\geqslant \limsup_{x\to\infty}\frac{\mathrm{P}(Z_0>2x,|Z_{-1}|\leqslant x,|Z_1|\leqslant -x)}{\mathrm{P}(|X_0|>x)} \\
&= \limsup_{x\to\infty}\frac{\mathrm{P}(Z_0>2x,|Z_{-1}|\leqslant x,|Z_1|\leqslant -x)}{\mathrm{P}(|X_0|>x)} \\
&= \limsup_{x\to\infty}\frac{\mathrm{P}(Z_0>2x)\mathrm{P}(|Z_{-1}|\leqslant x)\mathrm{P}(|Z_1|\leqslant -x)}{\mathrm{P}(|X_0|>x)} \\
&= \limsup_{x\to\infty}\frac{\mathrm{P}(Z_0>2x)}{\mathrm{P}(|Z_0|>2x)}\cdot\frac{\mathrm{P}(|Z_0|>2x)}{\mathrm{P}(|X_0|>2x)}\cdot\frac{\mathrm{P}(|X_0|>2x)}{\mathrm{P}(|X_0|>x)}\cdot[\mathrm{P}(|Z_1|\leqslant x)]^2 \\
&= \frac{1}{2}\cdot\frac{1}{2}\cdot 2^{-\alpha}\cdot 1>0.
\end{aligned}
$$

Therefore $\mathrm{P}(Y_0>0,Y_1<0)\geqslant \mathrm{P}(Y_0>1,Y_1<-1)>0$, i.e. the tail process (Y_n) has two values of the opposite sign with a positive probability. The remaining conditions from Theorem 2.15 hold (for details see Section 4.1). Thus the condition on the tail process as given in Theorem 2.15 can not be omitted.

Condition 2.14 is in general difficult to check. The next proposition gives one sufficient condition for Condition 2.14 to hold.

Proposition 2.19. *Let (X_n) be a strictly stationary sequence of regularly varying random variables with index $\alpha \in [1,2)$, and (a_n) a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. If the sequence (X_n) is ρ-mixing with*

$$\sum_{j \geqslant 0} \rho_{\lfloor 2^{j/3} \rfloor} < \infty,$$

then Condition 2.14 holds.

Proof. Let $\delta > 0$ be arbitrary. As in the proof of Theorem 2.15, define

$$I(u,n) = \mathrm{P}\bigg[\max_{1 \leqslant k \leqslant n} \bigg| \sum_{i=1}^{k} \bigg\{ \frac{X_i}{a_n} 1_{\big\{ \frac{|X_i|}{a_n} \leqslant u \big\}} - \mathrm{E}\bigg(\frac{X_i}{a_n} 1_{\big\{ \frac{|X_i|}{a_n} \leqslant u \big\}} \bigg) \bigg\} \bigg| > \delta \bigg].$$

Then from Corollary 2.1 in Peligrad [54] we obtain

$$I(u,n) \leqslant \delta^{-2} C \exp\bigg(8 \sum_{j=0}^{\lfloor \log_2 n \rfloor} \rho_{\lfloor 2^{j/3} \rfloor} \bigg) n\,\mathrm{E}\bigg[\bigg\{ \frac{X_1}{a_n} 1_{\big\{ \frac{|X_1|}{a_n} \leqslant u \big\}} - \mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\big\{ \frac{|X_1|}{a_n} \leqslant u \big\}} \bigg) \bigg\}^2 \bigg],$$

for some positive constant C. By assumption there exists a constant $L > 0$ such that, for all $n \in \mathbb{N}$,

$$\exp\bigg(8 \sum_{j=0}^{\lfloor \log_2 n \rfloor} \rho_{\lfloor 2^{j/3} \rfloor} \bigg) \leqslant L.$$

Therefore

$$\begin{aligned} I(u,n) &\leqslant CL\delta^{-2}\, n\,\mathrm{E}\bigg[\bigg(\frac{X_1}{a_n} 1_{\big\{ \frac{|X_1|}{a_n} \leqslant u \big\}} \bigg)^2 \bigg] \\ &= CL\delta^{-2} u^2 \cdot \frac{\mathrm{E}(X_1^2 1_{\{|X_1| \leqslant u a_n\}})}{(u a_n)^2 \mathrm{P}(|X_1| > u a_n)} \cdot n\mathrm{P}(|X_1| > u a_n). \end{aligned}$$

Now using Theorem 1.12 and the fact that X_1 is regularly varying (more precisely relation (1.7)), we obtain

$$\limsup_{n \to \infty} I(u,n) \leqslant CL\delta^{-2} \frac{\alpha}{2-\alpha} u^{2-\alpha}.$$

Since $2 - \alpha > 0$, we find $\lim_{u \downarrow 0} \limsup_{n \to \infty} I(u,n) = 0$, yielding Condition 2.14. $\square$

Remark 2.20. Theorem 2.15 becomes false if we replace the M_1 topology by Skorohod's J_1 topology: for finite order MA processes with at least two nonzero coefficients, Theorem 1 in Avram and Taqqu [3] shows that the sequence of partial sum stochastic processes V_n cannot have a weak limit in the J_1 topology.

The problem in our proof if we consider the J_1 topology is Lemma 2.8, which in this case does not hold. Fix $u > 0$ and define

$$\eta_n = \delta_{(\frac{1}{2}-\frac{1}{n},\,2u)} + \delta_{(\frac{1}{2},\,3u)}, \quad n \geqslant 3.$$

Then $\eta_n \xrightarrow{v} \eta$ as $n \to \infty$, where

$$\eta = \delta_{(\frac{1}{2},\,2u)} + \delta_{(\frac{1}{2},\,3u)}.$$

For $t_n = 1/2 - 1/n$ and every $\lambda \in \Delta$ we have

$$\psi^{(u)}(\eta_n)(t_n) = 2u \quad \text{and} \quad (\psi^{(u)}(\eta) \circ \lambda)(t_n) \in \{0, 5u\}.$$

Hence $\|\psi^{(u)}(\eta) \circ \lambda - \psi^{(u)}(\eta_n)\|_{[0,1]} \geqslant 2u$, and this implies $d_{J_1}(\psi^{(u)}(\eta), \psi^{(u)}(\eta_n)) \geqslant 2u$ for all $n \geqslant 3$, yielding that $\psi^{(u)}(\eta_n)$ does not converge to $\psi^{(u)}(\eta)$ as $n \to \infty$. Therefore $\psi^{(u)}$ is not continuous at η. Since clearly $\eta \in \Lambda$, we conclude that the summation functional $\psi^{(u)} \colon M_p([0,1] \times \overline{\mathbb{E}}_u) \to D[0,1]$ is not continuous on the set Λ, when $D[0,1]$ is endowed with Skorohod's J_1 topology.

Chapter 3

J_1 convergence in functional limit theorems

In this chapter we consider functional limit theorems in which the convergence is given with respect to Skorohod's J_1 topology. This happens in the i.i.d. case, the case of dependent random variables with isolated extremes and in the case when we do not deal with single random variables but with blocks of consecutive random variables of an appropriately chosen size.

3.1 The i.i.d. case

Functional limit theorems were at first studied for independent and identically distributed random variables. Note that if $(X_n)_{n\in\mathbb{N}}$ is an i.i.d. sequence of regularly varying random variables with index $\alpha \in (0,2)$ such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$, for some sequence of positive real numbers (a_n), then all conditions in Theorem 2.15 are satisfied. Indeed from Remark 1.15 it follows that the random process (X_n) is regularly varying with index α, while from the representation of the tail process for independent random variables in Example 1.18 we know that it almost surely has no two values of the opposite sign. The independence implies that (X_n) is strongly mixing, which by Proposition 1.34 further implies condition $\mathcal{A}'(a_n)$. Since the random variables X_i are independent and identically distributed we obtain that

$$\begin{aligned}
\mathrm{P}\Big(\max_{m\leqslant|i|\leqslant r_n}|X_i|>ua_n\,\Big|\,|X_0|>ua_n\Big) &\leqslant 2(r_n-m+1)\mathrm{P}(|X_1|>ua_n)\\
&= \frac{2(r_n-m+1)}{n}\cdot n\mathrm{P}(|X_1|>ua_n).
\end{aligned}$$

From the definition of a sequence (r_n) (see Definition 1.35) and the fact that X_1 is a regularly varying random variable, we obtain that, as $n\to\infty$,

$$\frac{r_n-m+1}{n}\to 0 \qquad \text{and} \qquad n\mathrm{P}(|X_1|>ua_n)\to u^{-\alpha}.$$

Hence

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathrm{P}\Big(\max_{m\leqslant|i|\leqslant r_n}|X_i|>ua_n\,\Big|\,|X_0|>ua_n\Big)=0,$$

and condition $\mathcal{AC}(a_n)$ holds. Since the random variables $X_1,X_2,\ldots$ are independent, (X_n) is ρ–mixing with $\rho_n=0$ for every $n\in\mathbb{N}$. Condition 2.14 now holds by Proposition 2.19. Therefore by Theorem 2.15 we have that the partial sum stochastic process V_n converges in distribution in $D[0,1]$ endowed with the M_1 topology, to an α–stable Leśvy process.

In this case, the M_1 convergence can be replaced by the J_1 convergence. This stronger result is well know in the literature, see Proposition 3.4 in Resnick [58] and Corollary 7.1 in Resnick [60]. These results are proved even in $D[0,\infty)$, thus with infinite time horizon. For completeness and to make an easier presentation of the results in the next sections, we give here the proof of the functional limit theorem for the i.i.d. case, but on $D[0,1]$ endowed with the J_1 topology. The proof, which we divide in several steps, follows the arguments given in Resnick [60], and its structure is very similar to the proof of Theorem 2.15. The first step is to establish a convergence in distribution of a sequence of time-space point processes

$$N_n=\sum_{i=1}^{n}\delta_{(i/n,X_i/a_n)}, \qquad n\in\mathbb{N},$$

similar to the one given in Theorem 1.36, with the difference that now the limiting process will be a Poisson random measure and the convergence will take place on the whole $[0,1]\times\mathbb{E}$. We start with two lemmas that will be useful in the sequel.

Lemma 3.1. *If (Z_n) and (T_n) are two sequences of random variables on the same probability space such that (Z_n) converges in distribution to some random variable Z and $T_n \xrightarrow{P} 0$, then $Z_nT_n \xrightarrow{P} 0$ as $n\to\infty$.*

For a proof of this result see for instance Chow and Teicher [20, p. 272]. Suppose now $H\colon \mathbb{R}\to(a,b)$ is a nondecreasing function on $\mathbb{R}$ with range (a,b), where $-\infty\leqslant a<b\leqslant\infty$. Define the *inverse* $H^{\leftarrow}\colon(a,b)\to\mathbb{R}$ of H as

$$H^{\leftarrow}(y)=\inf\{s : H(s)\geqslant y\}$$

(with the convention that the infimum of an empty set is $+\infty$). Then the following result holds (for a proof see for instance Proposition 0.1 in Resnick [59]).

Lemma 3.2. *If H_n, $n\geqslant 0$, are nondecreasing functions on $\mathbb{R}$ with range (a,b) and $H_n(x)\to H_0(x)$ for all $x\in C(H_0)$, then $H_n^{\leftarrow}(y)\to H_0^{\leftarrow}(y)$ for all $y\in(a,b)\cap C(H_0^{\leftarrow})$.*[1]

Now we are ready to describe the convergence of a sequence of point processes (N_n). We follow the proof of Theorem 6.3 in Resnick [60] (a different proof of this result is given in Proposition 3.1 in Resnick [58]).

Proposition 3.3. *Let (X_n) be a sequence of i.i.d. random variables such that, as $n\to\infty$,*

$$n\mathrm{P}\Big(\frac{X_1}{a_n}\in\cdot\Big)\xrightarrow{v}\mu(\cdot), \tag{3.1}$$

where (a_n) is a sequence of positive real numbers tending to ∞ and μ is a nonzero Radon measure on $(\mathbb{E},\mathcal{B}(\mathbb{E}))$. Then, as $n\to\infty$, $N_n\xrightarrow{d}N$ on $[0,1]\times\mathbb{E}$, where N is PRM($\mathbb{LEB}\times\mu$).

[1] Here $C(H)$ denotes the set of all $x\in\mathbb{R}$ such that H is finite and continuous at x.

Proof. From Example 1.24 (2) we know that the Laplace functional of the point process $N_n^* = \sum_{i=1}^n \delta_{X_i/a_n}$ is of the form

$$\Psi_{N_n^*}(f) = (\mathrm{E}e^{-f(X_1/a_n)})^n, \qquad f \in C_K^+(\mathbb{E}).$$

Therefore

$$\begin{aligned}\Psi_{N_n^*}(f) &= (\mathrm{E}e^{-f(X_1/a_n)})^n = \Big(1 - \frac{\mathrm{E}[n(1-e^{-f(X_1/a_n)})]}{n}\Big)^n \\ &= \Big(1 - \frac{\int_{\mathbb{E}}(1-e^{-f(x)})n\mathrm{P}(X_1/a_n \in dx)}{n}\Big)^n,\end{aligned}$$

and this, by (3.1) and Lemma 1.3 in Durrett [29, p. 80] (note that $1 - e^{-f} \in C_K^+(\mathbb{E})$), as $n \to \infty$ converges to

$$\exp\Big\{ - \int_{\mathbb{E}} (1 - e^{-f(x)})\, \mu(dx) \Big\},$$

the Laplace functional of $N^* = \mathrm{PRM}(\mu)$ (see Example 1.24 (3)). Theorem 1.23 now gives

$$N_n^* \xrightarrow{d} N^* = \mathrm{PRM}(\mu), \qquad \text{as } n \to \infty. \tag{3.2}$$

Suppose now $U_1, \ldots, U_n$ are i.i.d. random variables uniformly distributed on $(0,1)$ with order statistics

$$U_{1:n} \leqslant U_{2:n} \leqslant \ldots \leqslant U_{n:n},$$

which are independent of $\{X_i : i = 1, 2, \ldots\}$. Then by (3.2) and Lemma 1.25 we have that, as $n \to \infty$,

$$\sum_{i=1}^n \delta_{(U_i,\, X_i/a_n)} \xrightarrow{d} \mathrm{PRM}(\mathbb{LEB} \times \mu).$$

But since, from the independence of $\{U_i\}$ and $\{X_i\}$, we have that

$$\sum_{i=1}^n \delta_{(U_{i:n},\, X_i/a_n)} \stackrel{d}{=} \sum_{i=1}^n \delta_{(U_i,\, X_i/a_n)}$$

as random elements of $M_+([0,1]\times\mathbb{E})$, it holds that, as $n\to\infty$,

$$\sum_{i=1}^{n}\delta_{(U_{i:n},\,X_i/a_n)}\xrightarrow{d}\mathrm{PRM}(\mathbb{LEB}\times\mu). \tag{3.3}$$

If we prove that, as $n\to\infty$,

$$d_v\Big(\sum_{i=1}^{n}\delta_{(i/n,\,X_i/a_n)},\,\sum_{i=1}^{n}\delta_{(U_{i:n},\,X_i/a_n)}\Big)\xrightarrow{P}0, \tag{3.4}$$

where d_v is the metric given in (1.3), this and (3.3), by Slutsky's Theorem (see for instance Theorem 3.4 in Resnick [60]), will give that, as $n\to\infty$,

$$\sum_{i=1}^{n}\delta_{(i/n,\,X_i/a_n)}\xrightarrow{d}\mathrm{PRM}(\mathbb{LEB}\times\mu),$$

i.e. $N_n\xrightarrow{d}N$, and this proof will be completed.

By Proposition 1.26, for proving relation (3.4), it is enough to prove that for $f\in C_K^+([0,1]\times\mathbb{E})$, as $n\to\infty$,

$$\bigg|\sum_{i=1}^{n}f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)-\sum_{i=1}^{n}f\Big(U_{i:n},\frac{X_i}{a_n}\Big)\bigg|\xrightarrow{P}0. \tag{3.5}$$

Suppose the compact support of f is contained in $[0,1]\times\mathbb{E}_\delta$ for some $\delta>0$. Then the difference in (3.5) is bounded by

$$\begin{aligned}\sum_{i=1}^{n}&\bigg|f\Big(\frac{i}{n},\frac{X_i}{a_n}\Big)-f\Big(U_{i:n},\frac{X_i}{a_n}\Big)\bigg|1_{\{|X_i|>\delta a_n\}}\\&\leqslant\omega_{f,\delta}\bigg(\sup_{i\leqslant n}\Big|\frac{i}{n}-U_{i:n}\Big|\bigg)\sum_{i=1}^{n}1_{\{|X_i|>\delta a_n\}},\end{aligned}$$

where $\omega_{f,\delta}$ is the modulus of continuity of the function $f\big|_{[0,1]\times\overline{\mathbb{E}}_\delta}$, i.e.

$$\omega_{f,\delta}(\rho)=\sup\{|f(\mathbf{x})-f(\mathbf{y})|:\mathbf{x},\mathbf{y}\in[0,1]\times\overline{\mathbb{E}}_\delta,\,d_{[0,1]\times\mathbb{E}}(\mathbf{x},\mathbf{y})\leqslant\rho\}.^2$$

Define now the function $h\colon M_p(\mathbb{E})\to\mathbb{R}$ by

$$h(\eta)=\eta(\mathbb{E}_\delta),\qquad \eta\in M_p(\mathbb{E}).$$

[2] Recall the definition of the metric $d_{[0,1]\times\mathbb{E}}$ in Section 1.5.

Then h is continuous on the set

$$\Lambda^* = \{\eta \in M_p(\mathbb{E}) : \eta(\{\pm\delta, \pm\infty\}) = 0\}.$$

Indeed, take an arbitrary $\eta \in \Lambda^*$ and assume $\eta_n \xrightarrow{v} \eta$ in $M_p(\mathbb{E})$ as $n \to \infty$. Since the set $\mathbb{E}_\delta$ is relatively compact and $\eta(\partial\mathbb{E}_\delta) = 0$, from Theorem 1.4 we obtain that, as $n \to \infty$, $\eta_n(\mathbb{E}_\delta) \to \eta(\mathbb{E}_\delta)$, i.e. $h(\eta_n) \to h(\eta)$. Therefore h is continuous at η, and then, since $\eta \in \Lambda^*$ was arbitrary, on the set Λ^*. Since N^* is a Poisson random measure, it holds that $\mathrm{P}(N^*(\{\pm\delta, \pm\infty\}) = 0) = 1$, and this immediately implies that $\mathrm{P}(N^* \in \Lambda^*) = 1$. Thus, if D_h denotes the set of discontinuity points of h, we have

$$\mathrm{P}(N^* \in D_h) \leqslant \mathrm{P}(N^* \notin \Lambda^*) = 0.$$

The continuous mapping theorem (see Theorem 3.1 in Resnick [60]) applied to (3.2) then yields that, as $n \to \infty$,

$$N_n^*(\mathbb{E}_\delta) = h(N_n^*) \xrightarrow{d} h(N^*) = N^*(\mathbb{E}_\delta).$$

Therefore the sequence of random variables

$$\sum_{i=1}^{n} 1_{\{|X_i| > \delta a_n\}} = \sum_{i=1}^{n} \delta_{X_i/a_n}(\mathbb{E}_\delta) = N_n^*(\mathbb{E}_\delta)$$

converges in distribution to $N^*(\mathbb{E}_\delta)$ as $n \to \infty$. Hence by Lemma 3.1 it is enough to prove that, as $n \to \infty$,

$$\omega_{f,\delta}\left(\sup_{i \leqslant n} \left| \frac{i}{n} - U_{i:n} \right| \right) \xrightarrow{P} 0. \tag{3.6}$$

Since the function f restricted to the set $[0,1] \times \overline{\mathbb{E}}_\delta$ is uniformly continuous (since it is continuous on a compact set), it follows that $\omega_{f,\delta}(\rho) \to 0$ as $\rho \to 0$. Therefore for (3.6) to hold it is enough to prove that, as $n \to \infty$,

$$\sup_{i \leqslant n} \left| \frac{i}{n} - U_{i:n} \right| \xrightarrow{P} 0. \tag{3.7}$$

From the Glivenko-Cantelli theorem (see for instance Theorem 7.4 in Durrett [29, p. 59]) we know that, as $n \to \infty$,

$$\sup_{x\in[0,1]} \left| \frac{1}{n} \sum_{i=1}^{n} 1_{\{U_i \leqslant x\}} - x \right| \xrightarrow{a.s.} 0. \tag{3.8}$$

Put

$$\xi_n(x,\omega) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{U_i(\omega) \leqslant x\}} \quad \text{and} \quad \xi_0(x,\omega) = x$$

for all $x \in \mathbb{R}$ and $\omega \in \Omega$, where $(\Omega, \mathcal{F}, \mathrm{P})$ is the underlying probability space. Then from (3.8) it follows that, for all $x \in [0,1]$ and almost all ω,

$$\xi_n(x,\omega) \to \xi_0(x,\omega), \qquad \text{as } n \to \infty.$$

Since the functions $x \mapsto \xi_n(x,\omega)$ are nondecreasing, by Lemma 3.2 we have that, for almost all ω,

$$\xi_n^{\leftarrow}(x,\omega) \to \xi_0^{\leftarrow}(x,\omega) = x, \qquad \text{for all } x \in [0,1].$$

This gives monotone functions converging to a continuous limit and hence convergence is uniform on $[0,1]$ (see for instance Proposition 2.1 in Resnick [60]), i.e. for almost all ω,

$$\sup_{x\in[0,1]} |\xi_n^{\leftarrow}(x,\omega) - x| \to 0, \qquad \text{as } n \to \infty.$$

It is not hard to obtain

$$\xi_n^{\leftarrow}(x,\omega) = \sum_{i=1}^{n} U_{i:n}(\omega) 1_{(\frac{i-1}{n}, \frac{i}{n}]}(x), \qquad n \in \mathbb{N},$$

and

$$\sup_{i\leqslant n} \left| U_{i:n}(\omega) - \frac{i}{n} \right| \leqslant \sup_{x\in[0,1]} \left| \sum_{i=1}^{n} U_{i:n}(\omega) 1_{(\frac{i-1}{n}, \frac{i}{n}]}(x) - x \right|,$$

which yield that

$$\sup_{i\leqslant n} \left| U_{i:n} - \frac{i}{n} \right| \xrightarrow{a.s.} 0.$$

This immediately implies (3.7) and the proof is completed. □

For the proof of the functional limit theorem for the i.i.d. case with respect to Skorohod's J_1 topology we need a result on the continuity of the summation functional $\psi^{(u)}$ defined in Section 2.2. This result will have the same role in the i.i.d. case as Lemma 2.8 had in the proof of Theorem 2.15. Its proof is a slight modification of the one given in Resnick [60, Section 7.2.3] (one has to replace the state space $[0,\infty) \times \mathbb{E}$ with $[0,1] \times \overline{\mathbb{E}}_u$ which is straightforward), and is therefore here omitted.

Lemma 3.4. *The summation functional* $\psi^{(u)} \colon M_p([0,1] \times \overline{\mathbb{E}}_u) \to D[0,1]$ *is continuous on the set* $\Gamma = \Gamma_1 \cap \Gamma_2$*, when* $D[0,1]$ *is endowed with Skorohod's* J_1 *topology, where*

$$\Gamma_1 = \{\eta \in M_p([0,1] \times \overline{\mathbb{E}}_u) \, : \, \eta(\{0,1\} \times \mathbb{E}_u) = \eta([0,1] \times \{\pm\infty, \pm u\}) = 0\},$$

$$\Gamma_2 = \{\eta \in M_p([0,1] \times \overline{\mathbb{E}}_u) \, : \, \eta(\{t\} \times \overline{\mathbb{E}}_u) \leqslant 1 \text{ for all } t \in [0,1]\}.$$

Observe that the elements of Γ_2 have no two atoms with the same time coordinate. Equivalently, we can say that for every $\eta \in \Gamma_2$ no vertical line contains two points of η.

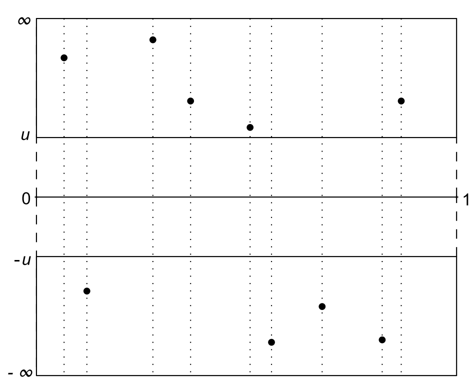

Figure 3.1: *An example of a point process belonging to the set* Γ.

Now we come to the final step in proving the functional limit theorem for the i.i.d. case. The proof of the following theorem follows the arguments presented in Resnick [60, Theorem 7.1 and Corollary 7.1].

Theorem 3.5. *Let $(X_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence of regularly varying random variables with index $\alpha \in (0,2)$, and let (a_n) be a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. Then the partial sum stochastic process*

$$V_n(t) = \sum_{k=1}^{[nt]} \frac{X_k}{a_n} - \lfloor nt \rfloor \mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\big\{ \frac{|X_1|}{a_n} \leqslant 1 \big\}} \bigg), \quad t \in [0,1], \tag{3.9}$$

satisfies

$$V_n \xrightarrow{d} V_0, \qquad n \to \infty,$$

in $D[0,1]$ endowed with the J_1 topology, where $V_0(\,\cdot\,)$ is an α–stable Lévy process with characteristic triple $(0, \mu, 0)$, where the measure μ is the vague limit of $n\mathrm{P}(X_1/a_n \in \cdot)$ as $n \to \infty$.

Proof. From Proposition 3.3 we know that, as $n \to \infty$,

$$\sum_{k=1}^{n} \delta_{(k/n, X_k/a_n)} \xrightarrow{d} N = \sum_{k} \delta_{(t_k,\, j_k)} = \mathrm{PRM}(\mathbb{LEB} \times \mu) \tag{3.10}$$

on $[0,1] \times \mathbb{E}$. Let $u \in (0,1)$ be arbitrary. Since by Theorem 1.6 (iii), $\mu(\{\pm u\}) = 0$, in a similar way as in the first part of the proof of Proposition 2.13 we obtain $\mathrm{P}(N \in \Gamma_1') = 1$, where

$$\Gamma_1' = \{ \eta \in M_p([0,1] \times \mathbb{E}) \, : \, \eta([0,1] \times \{\pm u\}) = 0 \}.$$

Since $\partial_{[0,1]\times\mathbb{E}}[0,1] \times \overline{\mathbb{E}}_u = [0,1] \times \{\pm u\}$, from Proposition 1.28 we obtain that the restriction map $T \colon M_p([0,1] \times \mathbb{E}) \to M_p([0,1] \times \overline{\mathbb{E}}_u)$ defined by

$$Tm = m\big|_{[0,1]\times\overline{\mathbb{E}}_u}$$

is continuous on the set Γ_1'. Thus from relation (3.10) and the continuous mapping theorem we get the restricted convergence

$$\sum_{k=1}^{n} 1_{\{|X_k| \geqslant u a_n\}} \delta_{(k/n, X_k/a_n)} \xrightarrow{d} \sum_{k} 1_{\{|j_k| \geqslant u\}} \delta_{(t_k,\, j_k)} \tag{3.11}$$

on $[0,1]\times\overline{\mathbb{E}}_u$. In a similar way as in the proof of Proposition 2.13 we obtain $\mathrm{P}(N\big|_{[0,1]\times\overline{\mathbb{E}}_u} \in \Gamma) = 1$. From this, relation (3.11) and the continuous mapping theorem we obtain that, as $n \to \infty$,

$$\sum_{k=1}^{\lfloor n\cdot\rfloor} \frac{X_k}{a_n} 1_{\{|X_k|>ua_n\}} \xrightarrow{d} \sum_{t_k \leqslant \cdot} j_k 1_{\{|j_k|>u\}} \tag{3.12}$$

in $D[0,1]$ under the J_1 metric.

Define the function $f_u \colon \mathbb{E} \to [0,\infty)$ with

$$f_u(x) = x \cdot 1_{\{x\,:\,u<|x|\leqslant 1\}}(x), \qquad x \in \mathbb{E}.$$

Then form the fact that $\mu_n(\cdot) := n\mathrm{P}(X_1/a_n \in \cdot) \xrightarrow{v} \mu(\cdot)$ as $n \to \infty$, using Proposition 1.2 (note that $\mu(D_{f_u}) = \mu(\{\pm u, \pm 1\}) = 0$ by Theorem 1.6 (iii)) we get $\int_{\mathbb{E}} f_u(x)\,\mu_n(dx) \to \int_{\mathbb{E}} f_u(x)\,\mu(dx)$, i.e.

$$n\,\mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\left\{u<\frac{|X_1|}{a_n}\leqslant 1\right\}}\bigg) \to \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx), \qquad \text{as } n \to \infty.$$

Therefore, for any $t \in [0,1]$, as $n \to \infty$,

$$\begin{aligned} \lfloor nt \rfloor \mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\left\{u<\frac{|X_1|}{a_n}\leqslant 1\right\}}\bigg) &= \frac{\lfloor nt \rfloor}{n} \cdot n\,\mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\left\{u<\frac{|X_1|}{a_n}\leqslant 1\right\}}\bigg) \\ &\to t \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx). \end{aligned}$$

This convergence is uniform in t and hence

$$\lfloor n\cdot\rfloor \mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\left\{u<\frac{|X_1|}{a_n}\leqslant 1\right\}}\bigg) \to (\,\cdot\,) \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx) \tag{3.13}$$

in the J_1 metric on $D[0,1]$.

Put

$$a_u = \int_{\{x\,:\,u<|x|\leqslant 1\}} x\,\mu(dx),$$

and define the function $x_u \colon [0,1] \to \mathbb{R}$ by $x^{(u)}(t) = ta_u$. The function $x^{(u)}$ is continuous, and hence it belongs to $D[0,1]$. Define now $h \colon D[0,1] \to D[0,1]$ by $h(x) = x - x^{(u)}$.

Let show that h is continuous (with respect to the J_1 topology on $D[0,1]$). Take an arbitrary $x \in D[0,1]$ and assume $d_{J_1}(x_n, x) \to 0$ as $n \to \infty$. Then from the first statement in Proposition 2.2 it follows that there exists a sequence of functions (λ_n) in Δ such that, as $n \to \infty$,

$$\|\lambda_n - e\|_{[0,1]} \to 0 \quad \text{and} \quad \|x_n \circ \lambda_n - x\|_{[0,1]} \to 0.$$

Then it holds that

$$\|h(x_n) \circ \lambda_n - h(x)\|_{[0,1]} \leqslant \|x_n \circ \lambda_n - x\|_{[0,1]} + |a_u| \cdot \|\lambda - e\|_{[0,1]} \to 0.$$

Another application of Proposition 2.2 now gives that $d_{J_1}(h(x_n), h(x)) \to 0$ as $n \to \infty$, showing the function h is continuous at x. Since x was chosen arbitrary, h is continuous on the whole $D[0,1]$. Hence by the continuous mapping theorem from (3.12) we obtain, as $n \to \infty$,

$$\widetilde{V}_n^{(u)}(\,\cdot\,) := \sum_{k=1}^{\lfloor n \cdot \rfloor} \frac{X_k}{a_n} 1_{\left\{\frac{|X_k|}{a_n} > u\right\}} - (\,\cdot\,) a_u \xrightarrow{d} V_0^{(u)}(\,\cdot\,) := \sum_{t_k \leqslant \cdot} j_k 1_{\{|j_k| > u\}} - (\,\cdot\,) a_u \tag{3.14}$$

in $D[0,1]$ under the J_1 metric. Define

$$V_n^{(u)}(\,\cdot\,) := \sum_{k=1}^{\lfloor n \cdot \rfloor} \frac{X_k}{a_n} 1_{\left\{\frac{|X_k|}{a_n} > u\right\}} - \lfloor n \cdot \rfloor \mathrm{E}\biggl(\frac{X_1}{a_n} 1_{\left\{u < \frac{|X_1|}{a_n} \leqslant 1\right\}} \biggr).$$

Then in a similar way as relation (2.10) in the proof of Theorem 2.15 we obtain, for any $\delta > 0$,

$$\lim_{n \to \infty} \mathrm{P}[d_{J_1}(\widetilde{V}_n^{(u)}, V_n^{(u)}) > \delta] = 0.$$

From this, relation (3.14) and Slutsky's theorem (see Theorem 3.4 in Resnick [60]), we get

$$V_n^{(u)}(\,\cdot\,) \xrightarrow{d} V_0^{(u)}(\,\cdot\,), \qquad \text{as } n \to \infty, \tag{3.15}$$

in $D[0,1]$ under the J_1 metric. From the Lévy-Itô representation of a Lévy process (see Section 5.5.3 in Resnick [60], Section 2.5 in Kyprianou [44] or Theorem 19.2 in Sato

[63]), there exists a Lévy process $V_0(\,\cdot\,)$ with characteristic triple $(0, \mu, 0)$ such that, as $u \downarrow 0$,

$$\sup_{t\in[0,1]} |V_0^{(u)}(t) - V_0(t)| \xrightarrow{a.s.} 0,$$

Since uniform convergence implies Skorohod's J_1 convergence, we get

$$d_{J_1}(V_0^{(u)}(\,\cdot\,), V_0(\,\cdot\,)) \to 0$$

almost surely as $u \downarrow 0$, and hence since almost sure convergence implies convergence in distribution,

$$V_0^{(u)}(\,\cdot\,) \xrightarrow{d} V_0(\,\cdot\,), \qquad \text{as } u \to 0, \tag{3.16}$$

in $D[0,1]$ with the J_1 metric.

If we show that

$$\lim_{u\downarrow 0} \limsup_{n\to\infty} \mathrm{P}[d_{J_1}(V_n^{(u)}, V_n) > \delta] = 0$$

for any $\delta > 0$, then from (3.15), (3.16) and Theorem 3.5 in Resnick [60] we will have, as $n \to \infty$,

$$V_n(\,\cdot\,) \xrightarrow{d} V_0(\,\cdot\,)$$

in $D[0,1]$ with the J_1 metric. Since the J_1 metric on $D[0,1]$ is bounded above by the uniform metric on $D[0,1]$, it suffices to show that

$$\lim_{u\downarrow 0} \limsup_{n\to\infty} \mathrm{P}\bigg(\sup_{t\in[0,1]} |V_n^{(u)}(t) - V_n(t)| > \delta \bigg) = 0. \tag{3.17}$$

Recalling the definitions, we have

$$\begin{aligned} &\mathrm{P}\bigg(\sup_{t\in[0,1]} |V_n^{(u)}(t) - V_n(t)| > \delta \bigg) \\ &\quad = \mathrm{P}\bigg[\sup_{t\in[0,1]} \bigg| \sum_{i=1}^{\lfloor nt \rfloor} \frac{X_i}{a_n} 1_{\big\{ \frac{|X_i|}{a_n} \leqslant u \big\}} - \lfloor nt \rfloor \mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\big\{ \frac{|X_1|}{a_n} \leqslant u \big\}} \bigg) \bigg| > \delta \bigg] \\ &\quad = \mathrm{P}\bigg[\max_{1\leqslant k\leqslant n} \bigg| \sum_{i=1}^{k} \bigg\{ \frac{X_i}{a_n} 1_{\big\{ \frac{|X_i|}{a_n} \leqslant u \big\}} - \mathrm{E}\bigg(\frac{X_i}{a_n} 1_{\big\{ \frac{|X_i|}{a_n} \leqslant u \big\}} \bigg) \bigg\} \bigg| > \delta \bigg]. \end{aligned}$$

By Kolmogorov's inequality (for example see Theorem 8.2 in Durrett [29, p. 62]) and the i.i.d. property of the random variables X_n, this has upper bound

$$
\begin{aligned}
&\leqslant \delta^{-2}\mathrm{Var}\bigg(\sum_{i=1}^{n}\frac{X_i}{a_n}1_{\big\{\frac{|X_i|}{a_n}\leqslant u\big\}}\bigg) = \delta^{-2}n\,\mathrm{Var}\bigg(\frac{X_1}{a_n}1_{\big\{\frac{|X_1|}{a_n}\leqslant u\big\}}\bigg)\\
&\leqslant \delta^{-2}n\,\mathrm{E}\bigg[\bigg(\frac{X_1}{a_n}\bigg)^2 1_{\big\{\frac{|X_1|}{a_n}\leqslant u\big\}}\bigg]\\
&= \delta^{-2}u^2\cdot\frac{\mathrm{E}(X_1^2 1_{\{|X_1|\leqslant ua_n\}})}{(ua_n)^2\mathrm{P}(|X_1|>ua_n)}\cdot n\mathrm{P}(|X_1|>ua_n).
\end{aligned}
$$

Now using Theorem 1.12 and the fact that X_1 is regularly varying with index $\alpha\in(0,2)$, we obtain

$$
\limsup_{n\to\infty}\mathrm{P}\bigg(\sup_{t\in[0,1]}|V_n^{(u)}(t)-V_n(t)|>\delta\bigg)\leqslant\delta^{-2}\,\frac{\alpha}{2-\alpha}\,u^{2-\alpha}.
$$

Letting $u\downarrow 0$, we easily get (3.17).

Finally, from Theorem 1.6 (ii) and Theorem 14.3. in Sato [63] it follows that the process $V_0(\,\cdot\,)$ is α–stable. □

As stated before, we presented here the detailed proof of the functional limit theorem for the i.i.d. case, which is well known in the literature, only for the sake of completeness and for an easier presentation of the results in the next sections.

Remark 3.6. In Theorem 3.5 the converse also holds. Precisely, let (X_n) be an i.i.d. sequence of random variables and let (a_n) be a sequence of positive real numbers such that $a_n\to\infty$ as $n\to\infty$. Define the measure μ for $x>0$ and $\alpha\in(0,2)$ by

$$
\mu((x,\infty])=px^{-\alpha},\qquad \mu((-\infty,-x])=qx^{-\alpha},
$$

where $p\in[0,1]$ and $q=1-p$. If $V_n\xrightarrow{d}V_0$ as $n\to\infty$, where $V_n(\,\cdot\,)$ is the partial sum stochastic process defined in (3.9) and $V_0(\,\cdot\,)$ is an α–stable Lévy process with characteristic triple $(0, \mu, 0)$, then the random variables X_n are regularly varying with index α and, as $n \to \infty$,

$$n\mathrm{P}\Big(\frac{X_1}{a_n} \in \cdot\Big) \xrightarrow{v} \mu(\cdot)$$

(see the necessity in Corollary 7.1 in Resnick [60]).

3.2 Isolated extremes

A regularly varying sequence of i.i.d. random variables has a tail process (Y_n) whose components, except Y_0, are zeros (see Example 1.18). Hence its extremes are isolated. A natural generalization of the functional limit theorem described in the previous section is the one that involves dependent random variables with the same tail process as in the i.i.d. case. One condition that assures this is the dependence condition D' as given in Davis [22]. Functional limit theorems for processes with isolated extremes can be found in Leadbetter and Rootzén [45] and Tyran-Kamińska [67]. We give here a shortened proof of the functional limit theorem for such processes for the sake of completeness and to make an illustration how the techniques used in the previous section can be applied to one class of dependent random variables. The emphasis will be on the convergence of point processes N_n^* to a Poisson random measure, as described in Balan and Louhichi [5].

Suppose (X_n) is a strictly stationary and strongly mixing sequence of regularly varying random variables with index $\alpha \in (0,2)$ that satisfies condition D', i.e.

$$\lim_{k\to\infty} \limsup_{n\to\infty} n \sum_{i=1}^{\lfloor n/k \rfloor} \mathrm{P}\bigg(\frac{|X_0|}{a_n} > x, \frac{|X_i|}{a_n} > x \bigg) = 0, \quad \text{for all } x > 0,$$

where (a_n) is a sequence of positive real numbers such that $n\mathrm{P}(|X_0| > a_n) \to 1$ as $n \to \infty$. It is straightforward to see that condition D' implies the following condition

$$\lim_{n\to\infty} n \sum_{i=1}^{r_n} \mathrm{P}\bigg(\frac{|X_0|}{a_n} > x, \frac{|X_i|}{a_n} > x \bigg) = 0, \quad \text{for all } x > 0, \tag{3.18}$$

where (r_n) is any sequence of positive integers such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$.

The functional limit theorem in this case is given in the following result. As in Theorem 2.15, here we will need to assume Condition 2.14 if $\alpha \in [1, 2)$.

Theorem 3.7. *Let (X_n) be a strictly stationary and strongly mixing sequence of regularly varying random variables with index $\alpha \in (0, 2)$, that satisfies condition (3.18). If $\alpha \in [1, 2)$, also suppose that Condition 2.14 holds. Then the partial sum stochastic process*

$$V_n(t) = \sum_{k=1}^{[nt]} \frac{X_k}{a_n} - \lfloor nt \rfloor \mathrm{E}\bigg(\frac{X_1}{a_n} 1_{\big\{ \frac{|X_1|}{a_n} \leqslant 1 \big\}} \bigg), \quad t \in [0, 1],$$

satisfies

$$V_n \xrightarrow{d} V_0, \qquad n \to \infty,$$

in $D[0, 1]$ endowed with the J_1 topology, where $V_0(\,\cdot\,)$ is an α–stable Lévy process with characteristic triple $(0, \mu, 0)$, where the measure μ is the vague limit of $n\mathrm{P}(X_1/a_n \in \cdot)$ as $n \to \infty$.

Proof. Recall $N_n^* = \sum_{i=1}^{n} \delta_{X_i/a_n}$. Put $k_n = \lfloor n/r_n \rfloor$ and define

$$\widetilde{N}_n = \sum_{i=1}^{k_n} \widetilde{N}_{r_n,\,i},$$

where $\widetilde{N}_{r_n,i}$, $i = 1, \ldots, k_n$, are i.i.d. point processes distributed as $N_{r_n}^*$. The strong mixing condition implies condition $\mathcal{A}(a_n)$ from Davis and Hsing [24]: for every $f \in C_K^+(\mathbb{E})$, as $n \to \infty$,

$$\mathrm{E} \exp\bigg\{ -\sum_{i=1}^{n} f\Big(\frac{X_i}{a_n}\Big) \bigg\} - \bigg[\mathrm{E} \exp\bigg\{ -\sum_{i=1}^{r_n} f\Big(\frac{X_i}{a_n}\Big) \bigg\} \bigg]^{k_n} \to 0.$$

This condition implies that N_n^* converges in distribution if and only if $\widetilde{N}_n$ does, and in that case they have the same limit. Let show that $N_n^* \xrightarrow{d} N^* = \mathrm{PRM}(\mu)$. It suffices to show, by Theorem 1.23, that for every $f \in C_K^+(\mathbb{E})$, as $n \to \infty$,

$$\Psi_{N_n^*}(f) \to \Psi_{N^*}(f).$$

For $m \leqslant n$ put

$$\Psi_{m,n}(f) = \mathrm{E} \exp\Big\{ - \sum_{i=1}^{m} f\Big(\frac{X_i}{a_n}\Big)\Big\}.$$

Then $\Psi_{N_n^*}(f) = \Psi_{n,n}(f)$. Since $\widetilde{N}_{r_n,i}$, $i = 1, \ldots, k_n$, are i.i.d. we have

$$\Psi_{\widetilde{N}_n}(f) = (\Psi_{r_n,n}(f))^{k_n}. \tag{3.19}$$

In a similar way as in the proof of Theorem 2.6 in Balan and Louhichi [5], using condition (3.18), stationarity and the regular variation property, we obtain that for every $f \in C_K^+(\mathbb{E})$, as $n \to \infty$,

$$k_n(1 - \Psi_{r_n,n}(f)) - n(1 - \Psi_{1,n}(f)) \to 0. \tag{3.20}$$

For $f \in C_K^+(\mathbb{E})$ the function $h \colon \mathbb{E} \to [0, \infty)$ defined by $h(x) = 1 - e^{-f(x)}$ is also in $C_K^+(\mathbb{E})$. Therefore the vague convergence $n\mathrm{P}(X_1/a_n \in \cdot) \xrightarrow{v} \mu(\cdot)$ implies $\int n\, h(X_1/a_n)\, d\mathrm{P} \to \int h(x)\, \mu(dx)$ as $n \to \infty$, i.e.

$$n(1 - \Psi_{1,n}(f)) = n\Big(1 - \mathrm{E} e^{-f\left(\frac{X_1}{a_n}\right)}\Big) \to \int_{\mathbb{E}} (1 - e^{-f(x)})\, \mu(dx). \tag{3.21}$$

From relations (3.20) and (3.21) we immediately get

$$\lim_{n \to \infty} k_n(1 - \Psi_{r_n,n}(f)) = \int_{\mathbb{E}} (1 - e^{-f(x)})\, \mu(dx).$$

From this using Lemma 1.3 in Durrett [29, p. 80], since $k_n \to \infty$ as $n \to \infty$, it follows that, as $n \to \infty$,

$$(\Psi_{r_n,n}(f))^{k_n} = \Big(1 - \frac{k_n(1 - \Psi_{r_n,n}(f))}{k_n}\Big)^{k_n} \to \exp\Big(- \int_{\mathbb{E}} (1 - e^{-f(x)})\, \mu(dx)\Big).$$

Hence, by (3.19), as $n \to \infty$,

$$\Psi_{\widetilde{N}_n}(f) \to \exp\Big(- \int_{\mathbb{E}} (1 - e^{-f(x)})\, \mu(dx) \Big).$$

Since N_n^* and $\widetilde{N}_n$ converge in distribution to the same limit, we have that, as $n \to \infty$,

$$\Psi_{N_n^*}(f) \to \exp\Big(- \int_{\mathbb{E}} (1 - e^{-f(x)})\, \mu(dx) \Big).$$

Since the limit is the Laplace functional of $N^* = \mathrm{PRM}(\mu)$ (see Example 1.24 (3)), we conclude that

$$N_n^* \xrightarrow{d} N^* = \mathrm{PRM}(\mu).$$

Now repeating the proof of Proposition 3.3 from relation (3.2) we obtain that, as $n \to \infty$,

$$N_n = \sum_{i=1}^{n} \delta_{(i/n, X_i/a_n)} \xrightarrow{d} N = \mathrm{PRM}(\mathbb{LEB} \times \mu).$$

This is in fact relation (3.10) in the proof of Theorem 3.5. We can repeat that proof here almost till the end. The only difference is that in proving (3.17) we can not use Kolmogorov's inequality (since our random variables X_n are not independent), but instead we proceed as at the end of the proof of Theorem 2.15 (so we use the arguments that were used in the proof of relation (2.14)). Therefore we conclude that $V_n \xrightarrow{d} V_0$ as $n \to \infty$, in $D[0,1]$ endowed with the J_1 topology. $\square$

Remark 3.8. Since the J_1 convergence implies the M_1 convergence, it holds that the process V_n, under the same conditions as in Theorem 3.7, converges in distribution to V_0 as $n \to \infty$, in $D[0,1]$ endowed with the M_1 topology.

3.3 Functional limit theorem with different partial sum process

As stated in Remark 2.20, we can not replace the M_1 topology in Theorem 2.15 by the J_1 topology. But if we alter the definition of the partial sum process in an appropriate way, we shall be able to recover the J_1 convergence for certain mixing sequences.

Let (X_n) be a strictly stationary sequence of regularly varying random variables with index $\alpha \in (0,2)$, and let (a_n) be a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. Define

$$\nu(x,\infty) = c_+ x^{-\alpha} \qquad \text{and} \qquad \nu(-\infty,-x) = c_- x^{-\alpha}, \qquad x > 0,$$

for some $c_+, c_- \geqslant 0$. These relations determine a Lévy measure ν which can then be written as

$$\nu(dx) = \big(c_+ \alpha x^{-\alpha-1} 1_{(0,\infty)}(x) + c_- \alpha (-x)^{-\alpha-1} 1_{(-\infty,0)}(x)\big)\,dx. \tag{3.22}$$

Let

$$S_m = \sum_{k=1}^{m} X_k, \qquad m \in \mathbb{N}.$$

In the sequel we will need that for every $x > 0$ the following large deviation relations

$$\begin{aligned} k_n \mathrm{P}(S_{r_n} > x a_n) &\to \nu(x,\infty), \\ k_n \mathrm{P}(S_{r_n} < -x a_n) &\to \nu(-\infty,-x), \end{aligned} \tag{3.23}$$

as $n \to \infty$, hold. Here (r_n) is a sequence of positive integers such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$, and $k_n = \lfloor n/r_n \rfloor$.

Remark 3.9. Some sufficient conditions for relations in (3.23) to hold are given in Bartkiewicz et al. [6] and Davis and Hsing [24]. We list here the conditions from [6].

1. The process (X_n) is regularly varying with index $\alpha \in (0,2)$.

2. For every $x \in \mathbb{R}$, as $n \to \infty$,

$$\big|\varphi_n(x) - (\varphi_{nr_n}(x))^{k_n}\big| \to 0,$$

where $\varphi_{nj}(x) = \mathrm{E}e^{ixa_n^{-1}S_j}$, $j = 1, 2, \ldots$, and $\varphi_n(x) = \varphi_{nn}(x)$.

3. For every $x \in \mathbb{R}$,

$$\lim_{d\to\infty} \limsup_{n\to\infty} \frac{n}{r_n} \sum_{j=d+1}^{r_n} \mathrm{E}\big|\overline{xa_n^{-1}(S_j - S_d)} \cdot \overline{xa_n^{-1}X_1}\big| = 0,$$

where for an arbitrary random variable Z we put $\overline{Z} = (Z \wedge 2) \vee (-2)$.

4. Assume the limits

$$\lim_{n\to\infty} n\,\mathrm{P}(S_d > a_n) = b_+(d) \quad \text{and} \quad \lim_{n\to\infty} n\,\mathrm{P}(S_d \leqslant -a_n) = b_-(d), \qquad d \in \mathbb{N},$$

$$\lim_{d\to\infty} (b_+(d) - b_+(d-1)) = c_+ \quad \text{and} \quad \lim_{d\to\infty} (b_-(d) - b_-(d-1)) = c_-$$

exists.

5. For $\alpha > 1$ assume $\mathrm{E}X_1 = 0$ and for $\alpha = 1$,

$$\lim_{d\to\infty} \limsup_{n\to\infty} n\left|\mathrm{E}(\sin(a_n^{-1}S_d))\right| = 0.$$

If these conditions hold then the relations in (3.23) hold (see relation (3.6) in [6]).

By Lemma 6.1 in Resnick [60], (3.23) is equivalent to

$$k_n \mathrm{P}\left(\frac{S_{r_n}}{a_n} \in \cdot\right) \xrightarrow{v} \nu(\,\cdot\,), \qquad \text{as } n \to \infty. \tag{3.24}$$

In the sequel we assume relation (3.24) holds. Define

$$S_{r_n}^{k,\,n} = X_{(k-1)r_n+1} + \ldots + X_{kr_n}, \qquad k,\, n \in \mathbb{N}$$

(note $S_{r_n}^{1,\,n} = S_{r_n}$).

Lemma 3.10. *Let $\alpha \in (0,1)$ and assume relation (3.24) holds. Then for any $u > 0$,*

$$\lim_{n\to\infty} k_n \mathrm{E}\left(\frac{|S_{r_n}|}{a_n} 1_{\left\{\frac{|S_{r_n}|}{a_n} \leqslant u\right\}}\right) = \int_{|x|\leqslant u} |x|\,\nu(dx). \tag{3.25}$$

Proof. Fix $u > 0$. Define

$$\nu_n(\,\cdot\,) = k_n \mathrm{P}(a_n^{-1} S_{r_n} \in \cdot\,), \qquad n \in \mathbb{N},$$

and

$$f_\delta(x) = |x| 1_{B(\delta,\,u)}(x), \qquad x \in \mathbb{E},\ \delta \in (0,u),$$

where $B(\delta,\,u) = \{x \in \mathbb{E} : \delta < |x| \leqslant u\}$. By relation (3.24), $\nu_n \xrightarrow{v} \nu$ as $n \to \infty$, and this with $\nu(\partial B(\delta,u)) = 0$ yield

$$\int_{\mathbb{E}} f_\delta(x)\, \nu_n(dx) \to \int_{\mathbb{E}} f_\delta(x)\, \nu(dx), \tag{3.26}$$

as $n \to \infty$ (see Proposition 1.2). Define

$$f(x) = |x| 1_{B(u)}(x), \qquad x \in \mathbb{E},$$

where $B(r) = \{x \in \mathbb{E} : |x| \leqslant r\}$. It follows

$$\begin{aligned}
\left| \int_{\mathbb{E}} f(x)\, \nu_n(dx) - \int_{\mathbb{E}} f(x)\, \nu(dx) \right| \ &\leq \ \left| \int_{B(\delta)} f(x)\, \nu_n(dx) - \int_{B(\delta)} f(x)\, \nu(dx) \right| \\
&\quad + \left| \int_{B(\delta)^c} f(x)\, \nu_n(dx) - \int_{B(\delta)^c} f(x)\, \nu(dx) \right| \\
&\leq \ \left| \int_{B(\delta)} f(x)\, \nu_n(dx) \right| + \left| \int_{B(\delta)} f(x)\, \nu(dx) \right| \\
&\quad + \left| \int_{B(\delta,\,u)} f(x)\, \nu_n(dx) - \int_{B(\delta,\,u)} f(x)\, \nu(dx) \right|,
\end{aligned} \tag{3.27}$$

for any $\delta \in (0,\,u)$. For the first term on the right hand side of (3.27) we have

$$\begin{aligned}
\left| \int_{B(\delta)} f(x)\, \nu_n(dx) \right| \ &= \ \int_{\mathbb{E}} |x| 1_{B(\delta)}(x)\, \nu_n(dx) = k_n \int \left| \frac{S_{r_n}}{a_n} \right| 1_{\{|S_{r_n}| \leqslant \delta a_n\}}\, d\mathrm{P} \\
&= \ k_n \mathrm{E}\left[\frac{|S_{r_n}|}{a_n} 1_{\{|S_{r_n}| \leqslant \delta a_n\}} \right] = k_n \mathrm{E}\left[\frac{|S_{r_n}|}{a_n} \underbrace{1_{\{|S_{r_n}| \leqslant \delta a_n\}}}_{\leqslant 1} 1_{\{\cap_{j=1}^{r_n} \{|X_j| \leqslant \delta a_n\}\}} \right] \\
&\quad + k_n \mathrm{E}\left[\underbrace{\frac{|S_{r_n}|}{a_n} 1_{\{|S_{r_n}| \leqslant \delta a_n\}}}_{\leqslant \delta} 1_{\{\cup_{j=1}^{r_n} \{|X_j| > \delta a_n\}\}} \right],
\end{aligned}$$

which implies it is bounded above by

$$
\begin{aligned}
&\leqslant \; k_n \mathrm{E}\bigg[\frac{\sum_{j=1}^{r_n}|X_j|}{a_n} 1_{\{\cap_{j=1}^{r_n}\{|X_j|\leqslant \delta a_n\}\}}\bigg] + k_n \delta \mathrm{P}\bigg(\bigcup_{j=1}^{r_n}\{|X_j| > \delta a_n\}\bigg) \\
&\leqslant \; k_n \sum_{j=1}^{r_n} \mathrm{E}\bigg[\frac{|X_j|}{a_n} 1_{\{|X_j|\leqslant \delta a_n\}}\bigg] + k_n \delta \sum_{j=1}^{r_n} \mathrm{P}(|X_j| > \delta a_n) \\
&= \; k_n r_n \mathrm{E}\bigg[\frac{|X_1|}{a_n} 1_{\{|X_1|\leqslant \delta a_n\}}\bigg] + k_n r_n \delta \mathrm{P}(|X_1| > \delta a_n) \\
&= \; \delta \cdot \frac{k_n r_n}{n} \cdot n\mathrm{P}(|X_1| > \delta a_n) \cdot \bigg[\frac{\mathrm{E}[|X_1| 1_{\{|X_1|\leqslant \delta a_n\}}]}{\delta a_n \mathrm{P}(|X_1| > \delta a_n)} + 1\bigg].
\end{aligned}
\tag{3.28}
$$

From the definition of sequences (r_n) and (k_n) it follows

$$\frac{k_n r_n}{n} \to 1, \qquad \text{as } n \to \infty.$$

Since X_1 is a regularly varying random variable with index α, it follows immediately

$$n\mathrm{P}(|X_1| > \delta a_n) \to \delta^{-\alpha}, \qquad \text{as } n \to \infty.$$

By Theorem 1.12 it holds that

$$\lim_{n\to\infty} \frac{\mathrm{E}[|X_1| 1_{\{|X_1|\leqslant \delta a_n\}}]}{\delta a_n \mathrm{P}(|X_1| > \delta a_n)} = \frac{\alpha}{1-\alpha}.$$

Now from (3.28) we get

$$\limsup_{n\to\infty} \bigg| \int_{B(\delta)} f(x)\, \nu_n(dx) \bigg| \leqslant \delta^{1-\alpha} \bigg(\frac{\alpha}{1-\alpha} + 1 \bigg),$$

and therefore, since $\alpha \in (0,1)$,

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \bigg| \int_{B(\delta)} f(x)\, \nu_n(dx) \bigg| = 0. \tag{3.29}$$

By the representation of the measure ν in (3.22) we get

$$\int_{|x|\leqslant\delta} |x|\, \nu(dx) = (c_- + c_+) \frac{\alpha}{1-\alpha} \delta^{1-\alpha}.$$

Hence for the second term on the right hand side of (3.27) we have

$$\bigg| \int_{B(\delta)} f(x)\,\nu(dx) \bigg| = \int_{|x|\leqslant\delta} |x|\,\nu(dx) \to 0, \qquad \text{as } \delta \to 0. \tag{3.30}$$

From (3.26) we get for the third term on the right hand side of (3.27)

$$\begin{aligned} \bigg| \int_{B(\delta,\,u)} f(x)\,\nu_n(dx) - \int_{B(\delta,\,u)} f(x)\,\nu(dx) \bigg| &= \bigg| \int_{\mathbb{E}} f_\delta(x)\,\nu_n(dx) - \int_{\mathbb{E}} f_\delta(x)\,\nu(dx) \bigg| \\ &\to 0, \qquad \text{as } n \to \infty. \end{aligned} \tag{3.31}$$

Now from (3.27) using (3.29), (3.30) and (3.31) we obtain

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \bigg| \int_{\mathbb{E}} f(x)\,\nu_n(dx) - \int_{\mathbb{E}} f(x)\,\nu(dx) \bigg| = 0.$$

From this immediately follows

$$\int_{\mathbb{E}} f(x)\,\nu_n(dx) \to \int_{\mathbb{E}} f(x)\,\nu(dx), \qquad \text{as } n \to \infty,$$

i.e.

$$k_n \mathrm{E}\bigg(\frac{|S_{r_n}|}{a_n} 1_{\big\{ \frac{|S_{r_n}|}{a_n} \leqslant u \big\}} \bigg) \to \int_{|x|\leqslant u} |x|\,\nu(dx), \qquad \text{as } n \to \infty.$$

$\square$

The mixing condition appropriate for the main result in this section is given in the following definition.

Definition 3.11. *We say a strictly stationary sequence of random variables* (X_n) *satisfies the* ***mixing condition*** $\mathcal{A}''(a_n)$ *if there exists a sequence of positive integers* (r_n) *such that* $r_n \to \infty$ *and* $r_n/n \to 0$ *as* $n \to \infty$*, and such that for every* $f \in C_K^+(\mathbb{E})$*, denoting* $k_n = \lfloor n/r_n \rfloor$*, as* $n \to \infty$*,*

$$\mathrm{E}\exp\bigg(-\sum_{k=1}^{k_n} f(a_n^{-1} S_{r_n}^{k,\,n}) \bigg) - \Big(\mathrm{E}\exp(-f(a_n^{-1} S_{r_n})) \Big)^{k_n} \to 0. \tag{3.32}$$

The mixing condition $\mathcal{A}''(a_n)$ holds if the process (X_n) is strongly mixing with geometric rate (see Proposition 3.14 and Remark 3.15 below). In case $\alpha \in [1,2)$, we will need to assume a condition similar to Condition 2.14.

Condition 3.12. There exists a sequence of positive integers (r_n) with $r_n \to \infty$ and $k_n = \lfloor n/r_n \rfloor \to \infty$ as $n \to \infty$, such that for all $\delta > 0$,

$$\lim_{u \downarrow 0} \limsup_{n \to \infty} \mathrm{P}\left[\max_{1 \leqslant j \leqslant k_n} \left| \sum_{k=1}^{j} \left(\frac{S_{r_n}^{k,n}}{a_n} 1_{\left\{ \frac{|S_{r_n}^{k,n}|}{a_n} \leqslant u \right\}} - \mathrm{E}\left(\frac{S_{r_n}^{k,n}}{a_n} 1_{\left\{ \frac{|S_{r_n}^{k,n}|}{a_n} \leqslant u \right\}} \right) \right) \right| > \delta \right] = 0.$$

Theorem 3.13. *Let (X_n) be a strictly stationary sequence of regularly varying random variables with index $\alpha \in (0,2)$, and let (a_n) be a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. Suppose there exists a sequence of positive integers (r_n) such that, as $n \to \infty$, $r_n \to \infty$, $k_n = \lfloor n/r_n \rfloor \to \infty$ and*

$$k_n \mathrm{P}\left(\frac{S_{r_n}}{a_n} \in \cdot \right) \xrightarrow{v} \nu(\cdot). \tag{3.33}$$

Suppose that the mixing condition $\mathcal{A}''(a_n)$ holds. If $\alpha \in [1,2)$, also suppose that Condition 3.12 holds. Then for a stochastic process defined by

$$W_n(t) = \sum_{k=1}^{\lfloor k_n t \rfloor} \frac{S_{r_n}^{k,n}}{a_n} - \lfloor k_n t \rfloor \mathrm{E}\left(\frac{S_{r_n}}{a_n} 1_{\left\{ \frac{|S_{r_n}|}{a_n} \leqslant 1 \right\}} \right), \quad t \in [0,1],$$

it holds that

$$W_n \xrightarrow{d} W_0, \qquad n \to \infty,$$

in $D[0,1]$ endowed with the J_1 topology, where $W_0(\,\cdot\,)$ is an α–stable Lévy process with characteristic triple $(0, \nu, 0)$.

Proof. Let, for any $n \in \mathbb{N}$, $(Z_{n,k})_k$ be a sequence of i.i.d. random variables such that $Z_{n,1} \stackrel{d}{=} S_{r_n}$. By relation (3.33) we have

$$k_n \mathrm{P}\left(\frac{Z_{n,1}}{a_n} \in \cdot \right) \xrightarrow{v} \nu(\cdot), \qquad \text{as } n \to \infty. \tag{3.34}$$

Proposition 1.27 then implies, as $n \to \infty$,

$$\widetilde{\xi}_n := \sum_{k=1}^{k_n} \delta_{a_n^{-1} Z_{n,k}} \xrightarrow{d} \mathrm{PRM}(\nu) \tag{3.35}$$

on $\mathbb{E}$. For any $n \in \mathbb{N}$ define a point process

$$\xi_n = \sum_{k=1}^{k_n} \delta_{a_n^{-1} S_{r_n}^{k,\,n}}.$$

For any $f \in C_K^+(\mathbb{E})$ we have

$$\begin{aligned}
\Psi_{\xi_n}(f) - \Psi_{\widetilde{\xi}_n}(f) &= \mathrm{E}\exp\bigg(-\sum_{k=1}^{k_n} f(a_n^{-1} S_{r_n}^{k,\,n}) \bigg) - \Big(\mathrm{E}\exp(-f(a_n^{-1} Z_{1,\,n})) \Big)^{k_n} \\
&= \mathrm{E}\exp\bigg(-\sum_{k=1}^{k_n} f(a_n^{-1} S_{r_n}^{k,\,n}) \bigg) - \Big(\mathrm{E}\exp(-f(a_n^{-1} S_{r_n})) \Big)^{k_n}.
\end{aligned}$$

Hence, the mixing condition $\mathcal{A}''(a_n)$ implies $\Psi_{\xi_n}(f) - \Psi_{\widetilde{\xi}_n}(f) \to 0$ as $n \to \infty$. Relation (3.35) and Theorem 1.23 then imply, as $n \to \infty$,

$$\sum_{k=1}^{k_n} \delta_{a_n^{-1} S_{r_n}^{k,\,n}} \xrightarrow{d} \mathrm{PRM}(\nu).$$

This corresponds to relation (3.2) in the proof of Proposition 3.3. Now using the same technique as in that proof we obtain that, as $n \to \infty$,

$$\sum_{k=1}^{k_n} \delta_{(k/k_n,\, a_n^{-1} S_{r_n}^{k,\,n})} \xrightarrow{d} \mathrm{PRM}(\mathbb{LEB} \times \nu) \tag{3.36}$$

on $[0,1] \times \mathbb{E}$. This relation corresponds to relation (3.10) in the proof of Theorem 3.5. We can repeat that proof in our case (we only need to put k_n instead of n in some places) until relation (3.16). It remains to prove

$$\lim_{u \downarrow 0} \limsup_{n \to \infty} \mathrm{P}\bigg(\sup_{t \in [0,1]} |W_n^{(u)}(t) - W_n(t)| > \delta \bigg) = 0,$$

for every $\delta > 0$, where

$$W_n^{(u)}(\,\cdot\,) := \sum_{k=1}^{\lfloor k_n \cdot \rfloor} \frac{S_{r_n}^{k,\,n}}{a_n} 1_{\big\{ \frac{|S_{r_n}^{k,\,n}|}{a_n} > u \big\}} - \lfloor k_n \cdot \rfloor \mathrm{E}\bigg(\frac{S_{r_n}}{a_n} 1_{\big\{ u < \frac{|S_{r_n}|}{a_n} \leqslant 1 \big\}} \bigg).$$

We have

$$\begin{aligned}
&\mathrm{P}\bigg(\sup_{t\in[0,1]}|W_n^{(u)}(t)-W_n(t)|>\delta\bigg)\\
&= \mathrm{P}\bigg[\max_{1\leqslant j\leqslant k_n}\bigg|\sum_{k=1}^{j}\bigg(\frac{S_{r_n}^{k,n}}{a_n}1_{\big\{\frac{|S_{r_n}^{k,n}|}{a_n}\leqslant u\big\}}-\mathrm{E}\bigg(\frac{S_{r_n}^{k,n}}{a_n}1_{\big\{\frac{|S_{r_n}^{k,n}|}{a_n}\leqslant u\big\}}\bigg)\bigg)\bigg|>\delta\bigg]. \qquad (3.37)
\end{aligned}$$

For $\alpha\in[1,2)$ this relation is simply Condition 3.12. Therefore it remains to show (3.37) for the case when $\alpha\in(0,1)$. Hence assume $\alpha\in(0,1)$. For arbitrary (and fixed) $\delta>0$ define

$$I(u,n)=\mathrm{P}\bigg[\max_{1\leqslant j\leqslant k_n}\bigg|\sum_{k=1}^{j}\bigg(\frac{S_{r_n}^{k,n}}{a_n}1_{\big\{\frac{|S_{r_n}^{k,n}|}{a_n}\leqslant u\big\}}-\mathrm{E}\bigg(\frac{S_{r_n}^{k,n}}{a_n}1_{\big\{\frac{|S_{r_n}^{k,n}|}{a_n}\leqslant u\big\}}\bigg)\bigg)\bigg|>\delta\bigg].$$

Using stationarity and Chebyshev's inequality we get the bound

$$\begin{aligned}
I(u,n) &\leqslant \mathrm{P}\bigg[\max_{1\leqslant j\leqslant k_n}\sum_{k=1}^{j}\bigg|\frac{S_{r_n}^{k,n}}{a_n}1_{\big\{\frac{|S_{r_n}^{k,n}|}{a_n}\leqslant u\big\}}-\mathrm{E}\bigg(\frac{S_{r_n}^{k,n}}{a_n}1_{\big\{\frac{|S_{r_n}^{k,n}|}{a_n}\leqslant u\big\}}\bigg)\bigg|>\delta\bigg]\\
&= \mathrm{P}\bigg[\sum_{k=1}^{k_n}\bigg|\frac{S_{r_n}^{k,n}}{a_n}1_{\big\{\frac{|S_{r_n}^{k,n}|}{a_n}\leqslant u\big\}}-\mathrm{E}\bigg(\frac{S_{r_n}^{k,n}}{a_n}1_{\big\{\frac{|S_{r_n}^{k,n}|}{a_n}\leqslant u\big\}}\bigg)\bigg|>\delta\bigg]\\
&\leqslant \delta^{-1}\mathrm{E}\bigg[\sum_{k=1}^{k_n}\bigg|\frac{S_{r_n}^{k,n}}{a_n}1_{\big\{\frac{|S_{r_n}^{k,n}|}{a_n}\leqslant u\big\}}-\mathrm{E}\bigg(\frac{S_{r_n}^{k,n}}{a_n}1_{\big\{\frac{|S_{r_n}^{k,n}|}{a_n}\leqslant u\big\}}\bigg)\bigg|\bigg]\\
&\leqslant 2\delta^{-1}\sum_{k=1}^{k_n}\mathrm{E}\bigg(\frac{|S_{r_n}^{k,n}|}{a_n}1_{\big\{\frac{|S_{r_n}^{k,n}|}{a_n}\leqslant u\big\}}\bigg)\\
&= 2\delta^{-1}k_n\mathrm{E}\bigg(\frac{|S_{r_n}|}{a_n}1_{\big\{\frac{|S_{r_n}|}{a_n}\leqslant u\big\}}\bigg).
\end{aligned}$$

Using Lemma 3.10 we obtain that

$$\begin{aligned}
\lim_{n\to\infty}k_n\mathrm{E}\bigg(\frac{|S_{r_n}|}{a_n}1_{\big\{\frac{|S_{r_n}|}{a_n}\leqslant u\big\}}\bigg) &= \int_{|x|\leqslant u}|x|\,\nu(dx)\\
&= (c_-+c_+)\frac{\alpha}{1-\alpha}u^{1-\alpha}\\
&\to 0, \qquad \text{as } u\to 0.
\end{aligned}$$

Hence

$$\lim_{u\downarrow 0}\limsup_{n\to\infty} I(u,n) = 0,$$

which completes the proof, with the note that the α–stability of the process $W_0(\,\cdot\,)$ follows from Theorem 14.3 in Sato [63] and the representation of the measure ν in (3.22). □

At the end of this section we give some sufficient conditions for the mixing condition $\mathcal{A}''(a_n)$ and Condition 3.12 to hold.

Proposition 3.14. *Suppose (X_n) is a strictly stationary sequence of regularly varying random variables with index $\alpha \in (0,2)$, and (a_n) a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. Assume relation (3.33) holds for some sequence of positive integers (r_n) such that $r_n \to \infty$ and $k_n = \lfloor n/r_n \rfloor \to \infty$ as $n \to \infty$, and $k_n = o(n^t)$ for some $0 < t < 1$. If the sequence (X_n) is strongly mixing with*

$$k_n \alpha_{l_n+1} \to 0, \qquad \text{as } n \to \infty, \tag{3.38}$$

where (α_n) is the sequence of α–mixing coefficients of (X_n) and (l_n) is a sequence of positive integers such that $l_n \to \infty$ as $n \to \infty$ and $l_n = o(n^q)$ for some $0 < q < \min\{1/\alpha, (1-t)/(1+\alpha)\}$, then the mixing condition $\mathcal{A}''(a_n)$ holds.

Proof. Let n be large enough such that $l_n < r_n$ (note that for large n it holds that $l_n < n^{1-t} < r_n$). We break $X_1, X_2, \ldots$ into blocks of r_n consecutive random variables. The last l_n variables in each block will be dropped. Then we shall show that doing so, the new blocks will be almost independent (as $n \to \infty$) and this will imply relation (3.32) for the new blocks. The error which occurs by cutting of the ends of the original blocks will be small, and this will imply condition (3.32) for the original blocks also.

Take an arbitrary $f \in C_K^+(\mathbb{E})$. Since its support is bounded away from 0, there exists some $r > 0$ such that $f(x) = 0$ for $|x| \leqslant r$, and since f is bounded, there exists some $M > 0$ such that $|f(x)| < M$ for all $x \in \mathbb{E}$. For all $k, n \in \mathbb{N}$ define

$$S_{r_n, l_n}^{k,n} = X_{kr_n - l_n + 1} + \ldots + X_{kr_n}.$$

$S_{r_n, l_n}^{k,n}$ is the sum of the last l_n random variables in the k-th block. By stationarity we have

$$S_{r_n}^{k,n} - S_{r_n, l_n}^{k,n} \stackrel{d}{=} S_{r_n}^{1,n} - S_{r_n, l_n}^{1,n} = S_{r_n - l_n}.$$

This and the following inequality

$$|\mathrm{E}gh - \mathrm{E}g\mathrm{E}h| \leqslant 4C_1C_2\alpha_m,$$

for a $\mathcal{F}_{-\infty}^{j}$ measurable function g and a $\mathcal{F}_{j+m}^{\infty}$ measurable function h such that $|g| \leqslant C_1$ and $|h| \leqslant C_2$ (see Lemma 1.2.1 in Lin and Lu [47]), applied k_n times, give

$$\begin{aligned} &\left| \mathrm{E}\exp\Bigg(- \sum_{k=1}^{k_n} f(a_n^{-1}S_{r_n}^{k,n} - a_n^{-1}S_{r_n,l_n}^{k,n}) \Bigg) - \Bigg(\mathrm{E}\exp(-f(a_n^{-1}S_{r_n - l_n})) \Bigg)^{k_n} \right| \\ &\qquad \leqslant 4k_n\alpha_{l_n+1}. \end{aligned} \tag{3.39}$$

Then

$$\begin{aligned} &\left| \mathrm{E}\exp\Bigg(- \sum_{k=1}^{k_n} f(a_n^{-1}S_{r_n}^{k,n}) \Bigg) - \Bigg(\mathrm{E}\exp(-f(a_n^{-1}S_{r_n})) \Bigg)^{k_n} \right| \\ &\quad \leqslant \left| \mathrm{E}\exp\Bigg(- \sum_{k=1}^{k_n} f(a_n^{-1}S_{r_n}^{k,n}) \Bigg) - \mathrm{E}\exp\Bigg(- \sum_{k=1}^{k_n} f(a_n^{-1}S_{r_n}^{k,n} - a_n^{-1}S_{r_n,l_n}^{k,n}) \Bigg) \right| \\ &\qquad + \left| \mathrm{E}\exp\Bigg(- \sum_{k=1}^{k_n} f(a_n^{-1}S_{r_n}^{k,n} - a_n^{-1}S_{r_n,l_n}^{k,n}) \Bigg) - \Bigg(\mathrm{E}\exp(-f(a_n^{-1}S_{r_n - l_n})) \Bigg)^{k_n} \right| \\ &\qquad + \left| \Bigg(\mathrm{E}\exp(-f(a_n^{-1}S_{r_n - l_n})) \Bigg)^{k_n} - \Bigg(\mathrm{E}\exp(-f(a_n^{-1}S_{r_n})) \Bigg)^{k_n} \right| \\ &\quad =: I_1(n) + I_2(n) + I_3(n). \end{aligned} \tag{3.40}$$

By Lemma 4.3 in Durrett [29] and stationarity we have

$$\begin{aligned} I_1(n) &\leqslant \mathrm{E}\bigg(\sum_{k=1}^{k_n} \big| e^{-f(a_n^{-1}S_{r_n}^{k,n})} - e^{-f(a_n^{-1}S_{r_n}^{k,n} - a_n^{-1}S_{r_n,l_n}^{k,n})} \big| \bigg) \\ &= k_n \mathrm{E} \big| e^{-f(a_n^{-1}S_{r_n})} - e^{-f(a_n^{-1}S_{r_n-l_n})} \big| \\ &= k_n \mathrm{E} \big| e^{-f(a_n^{-1}S_{r_n})} (1 - e^{f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})}) \big| \\ &\leqslant k_n \mathrm{E} \big| 1 - e^{f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})} \big|. \end{aligned}$$

It can be shown that for any $t > 0$ there exists a constant $C = C(t) > 0$ such that

$$|1 - e^{-x}| \leqslant C|x|, \qquad \text{for all } |x| < t.$$

Since for all $x, y \in \mathbb{E}$, $|f(x) - f(y)| < 2M$, there exists a positive constant C such that

$$I_1(n) \leqslant C k_n \mathrm{E} |f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})|. \tag{3.41}$$

Further, since $f(x) = 0$ for $|x| \leqslant r$, we have

$$\begin{aligned} &\mathrm{E}|f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})| \\ &\quad = \mathrm{E}\big[|f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})| 1_{\{a_n^{-1}|S_{r_n-l_n}| > r/2\}} 1_{\{a_n^{-1}|S_{r_n}| > r/4\}} \big] \\ &\qquad + \mathrm{E}\big[f(a_n^{-1}S_{r_n-l_n}) 1_{\{a_n^{-1}|S_{r_n-l_n}| > r/2\}} 1_{\{a_n^{-1}|S_{r_n}| \leqslant r/4\}} \big] \\ &\qquad + \mathrm{E}\big[f(a_n^{-1}S_{r_n}) 1_{\{a_n^{-1}|S_{r_n-l_n}| \leqslant r/2\}} 1_{\{a_n^{-1}|S_{r_n}| > r\}} \big] \\ &\quad \leqslant \mathrm{E}\big[|f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})| 1_{\{a_n^{-1}|S_{r_n-l_n}| > r/2\}} 1_{\{a_n^{-1}|S_{r_n}| > r/4\}} \big] \\ &\qquad + M\mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n} > \frac{r}{4} \bigg) + M\mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n} > \frac{r}{2} \bigg). \end{aligned} \tag{3.42}$$

Since the set $S = \{x \in \mathbb{E} : |x| > r/4\}$ is relatively compact and any continuous function on a compact set is uniformly continuous, it follows that for any $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in S$ such that $\rho(x, y) \leqslant \delta$, where ρ

is the metric on $\mathbb{E}$ defined in (1.1). If $|x| > r/2$, $|y| > r/4$ and $\mathrm{sign}(x) = \mathrm{sign}(y)$, then $x, y \in S$ and

$$\rho(x,y) = \frac{||x| - |y||}{|xy|} \leqslant \frac{8}{r^2}|x-y|. \tag{3.43}$$

Define $g_n(x,y) = |f(a_n^{-1}x) - f(a_n^{-1}y)|$ and let $\epsilon > 0$ be arbitrary. Then

$$\begin{aligned}
&\mathrm{E}\big[|f(a_n^{-1}S_{r_n}) - f(a_n^{-1}S_{r_n-l_n})|1_{\{a_n^{-1}|S_{r_n-l_n}|>r/2\}}1_{\{a_n^{-1}|S_{r_n}|>r/4\}}\big] \\
&\quad = \mathrm{E}\big[g_n(S_{r_n}, S_{r_n-l_n})1_{\{a_n^{-1}|S_{r_n-l_n}|>r/2,\, a_n^{-1}|S_{r_n}|>r/4\}}1_{\{\mathrm{sign}(S_{r_n-l_n})\neq \mathrm{sign}(S_{r_n})\}}\big] \\
&\qquad + \mathrm{E}\big[g_n(S_{r_n}, S_{r_n-l_n})1_{\{a_n^{-1}S_{r_n-l_n}>r/2,\, a_n^{-1}S_{r_n}>r/4\}}1_{\{a_n^{-1}|S_{r_n}-S_{r_n-l_n}|\leqslant \delta r^2/8\}}\big] \\
&\qquad + \mathrm{E}\big[g_n(S_{r_n}, S_{r_n-l_n})1_{\{a_n^{-1}S_{r_n-l_n}<-r/2,\, a_n^{-1}S_{r_n}<-r/4\}}1_{\{a_n^{-1}|S_{r_n}-S_{r_n-l_n}|\leqslant \delta r^2/8\}}\big] \\
&\qquad + \mathrm{E}\big[g_n(S_{r_n}, S_{r_n-l_n})1_{\{a_n^{-1}S_{r_n-l_n}>r/2,\, a_n^{-1}S_{r_n}>r/4\}}1_{\{a_n^{-1}|S_{r_n}-S_{r_n-l_n}|> \delta r^2/8\}}\big] \\
&\qquad + \mathrm{E}\big[g_n(S_{r_n}, S_{r_n-l_n})1_{\{a_n^{-1}S_{r_n-l_n}<-r/2,\, a_n^{-1}S_{r_n}<-r/4\}}1_{\{a_n^{-1}|S_{r_n}-S_{r_n-l_n}|> \delta r^2/8\}}\big].
\end{aligned}$$

By stationarity and relation (3.43) this is bounded above by

$$\begin{aligned}
&\leqslant 2M\mathrm{P}\bigg(\frac{|S_{r_n} - S_{r_n-l_n}|}{a_n} > \frac{3r}{4}\bigg) \\
&\qquad + \mathrm{E}\big[g_n(S_{r_n}, S_{r_n-l_n})1_{\{a_n^{-1}S_{r_n-l_n}>r/2\}}1_{\{a_n^{-1}S_{r_n}>r/4\}}1_{\{\rho(a_n^{-1}S_{r_n},\, a_n^{-1}S_{r_n-l_n})\leqslant\delta\}}\big] \\
&\qquad + \mathrm{E}\big[g_n(S_{r_n}, S_{r_n-l_n})1_{\{a_n^{-1}S_{r_n-l_n}<-r/2\}}1_{\{a_n^{-1}S_{r_n}<-r/4\}}1_{\{\rho(a_n^{-1}S_{r_n},\, a_n^{-1}S_{r_n-l_n})\leqslant\delta\}}\big] \\
&\qquad + 4M\mathrm{P}\bigg(\frac{|S_{r_n} - S_{r_n-l_n}|}{a_n} > \frac{\delta r^2}{8}\bigg) \\
&\leqslant 2M\mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n} > \frac{3r}{4}\bigg) + \epsilon\mathrm{P}\bigg(\frac{|S_{r_n}|}{a_n} > \frac{r}{4}\bigg) + 4M\mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n} > \frac{\delta r^2}{8}\bigg).
\end{aligned}$$

Therefore, from (3.41) and (3.42) we obtain

$$I_1(n) \leqslant 8MCk_n\mathrm{P}\bigg(\frac{|S_{l_n}|}{a_n} > \gamma\bigg) + \epsilon Ck_n\mathrm{P}\bigg(\frac{|S_{r_n}|}{a_n} > \frac{r}{4}\bigg), \tag{3.44}$$

where $\gamma = \min\{r/4, \delta r^2/8\} > 0$.

Since X_1 is regularly varying with index $\alpha \in (0,2)$, for any $x > 0$

$$\mathrm{P}(|X_1| > x) = x^{-\alpha} L(x),$$

where L is a slowly varying function (see Proposition 1.8). It also holds

$$a_n = n^{1/\alpha} L'(n),$$

where L' is a slowly varying function (see Remark 1.9). Therefore, taking an arbitrary $0 < s < \min\{\alpha,\, \alpha(1-t-q-\alpha q)/(1-\alpha q)\}$, we have

$$\begin{aligned} k_n \mathrm{P}\biggl(\frac{|S_{l_n}|}{a_n} > \gamma \biggr) &\leqslant k_n l_n \mathrm{P}(|X_1| > \gamma a_n / l_n) = k_n l_n \biggl(\frac{\gamma a_n}{l_n} \biggr)^{-\alpha} L \biggl(\frac{\gamma a_n}{l_n} \biggr) \\ &= k_n l_n \biggl(\frac{\gamma a_n}{l_n} \biggr)^{s-\alpha} \cdot c_n, \end{aligned}$$

where

$$c_n = \biggl(\frac{\gamma a_n}{l_n} \biggr)^{-s} L \biggl(\frac{\gamma a_n}{l_n} \biggr).$$

Since $a_n / l_n \to \infty$ as $n \to \infty$, by Proposition 1.3.6 in Bingham et al. [13] we have that $c_n \to 0$ as $n \to \infty$. Further

$$\begin{aligned} k_n l_n \biggl(\frac{\gamma a_n}{l_n} \biggr)^{s-\alpha} &= \frac{k_n (l_n)^{1+\alpha-s}}{\gamma^{\alpha-s} a_n^{\alpha-s}} = \biggl(\frac{l_n}{n^q} \biggr)^{1+\alpha-s} \cdot \frac{k_n}{n^t} \cdot \frac{n^t (n^q)^{1+\alpha-s}}{\gamma^{\alpha-s} n^{(\alpha-s)/\alpha} (L'(n))^{\alpha-s}} \\ &\leqslant \biggl(\frac{l_n}{n^q} \biggr)^{1+\alpha-s} \cdot \frac{k_n}{n^t} \cdot \frac{1}{\gamma^{\alpha-s} n^p (L'(n))^{\alpha-s}} \end{aligned}$$

where $p = (\alpha - s)/\alpha - t - (1 + \alpha - s)q$. It can easily be checked that $p > 0$. This and the fact that $l_n = o(n^q)$ and $k_n = o(n^t)$, by Proposition 1.3.6 in Bingham et al. [13], imply that $k_n l_n (\gamma a_n / l_n)^{s-\alpha} \to 0$ as $n \to \infty$. Hence

$$k_n \mathrm{P}\biggl(\frac{|S_{l_n}|}{a_n} > \gamma \biggr) \to 0, \quad \text{as } n \to \infty. \tag{3.45}$$

From relation (3.33) we obtain that, as $n \to \infty$,

$$k_n \mathrm{P}\biggl(\frac{|S_{r_n}|}{a_n} > \frac{r}{4} \biggr) \to \nu(\{x \in \mathbb{E} : |x| > r/4\}) =: A < \infty. \tag{3.46}$$

Thus from relations (3.44), (3.45) and (3.46) we obtain

$$\limsup_{n\to\infty} I_1(n) \leqslant AC\epsilon,$$

and since $\epsilon > 0$ is arbitrary, we have

$$\lim_{n\to\infty} I_1(n) = 0. \tag{3.47}$$

From (3.39) and the assumption that $k_n \alpha_{l_n+1} \to 0$ as $n \to \infty$, it follows immediately

$$\lim_{n\to\infty} I_2(n) = 0. \tag{3.48}$$

Using again Lemma 4.3 in Durrett [29] it follows

$$I_3(n) \leqslant k_n \mathrm{E}\Big| e^{-f(a_n^{-1} S_{r_n})} - e^{-f(a_n^{-1} S_{r_n - l_n})} \Big|.$$

Repeating the same procedure as for $I_1(n)$ we get

$$\lim_{n\to\infty} I_3(n) = 0. \tag{3.49}$$

Taking into account relations (3.47), (3.48) and (3.49), from (3.40) we obtain that, as $n \to \infty$,

$$\mathrm{E} \exp\bigg(- \sum_{k=1}^{k_n} f(a_n^{-1} S_{r_n}^{k,n}) \bigg) - \Big(\mathrm{E} \exp(-f(a_n^{-1} S_{r_n})) \Big)^{k_n} \to 0,$$

and this concludes the proof. $\square$

Remark 3.15. Relation (3.38) holds if (X_n) is strongly mixing with geometric rate, i.e. $\alpha_n \leqslant C\rho^n$ for some $\rho \in (0,1)$ and $C > 0$, and $l_n \sim n^r$ for some $r > 0$, i.e. $l_n/n^r \to 1$ as $n \to \infty$. Therefore if the sequence (X_n) is strongly mixing with geometric rate, then the mixing condition $\mathcal{A}''(a_n)$ holds.

Proposition 3.16. *Suppose (X_n) is a strictly stationary sequence of symmetric and regularly varying random variables with index of regular variation $\alpha \in [1, 2)$, and (a_n) a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. If the sequence (ρ_n) of ρ–mixing coefficients of (X_n) decreases to zero as $n \to \infty$ and*

$$\sum_{j \geqslant 0} \rho_{\lfloor 2^{j/3} \rfloor} < \infty, \tag{3.50}$$

then Condition 3.12 holds.

Proof. Let $n \in \mathbb{N}$ and $u > 0$ be arbitrary. Define

$$Z_k = Z_k(u, n) = \frac{S_{r_n}^{k,\,n}}{a_n} 1_{\left\{\frac{|S_{r_n}^{k,\,n}|}{a_n} \leqslant u\right\}} - \mathrm{E}\Big(\frac{S_{r_n}^{k,\,n}}{a_n} 1_{\left\{\frac{|S_{r_n}^{k,\,n}|}{a_n} \leqslant u\right\}}\Big), \qquad k \in \mathbb{N}.$$

Take an arbitrary $\delta > 0$ and as in the proof of Theorem 3.13 define

$$I(u, n) = \mathrm{P}\bigg[\max_{1 \leqslant j \leqslant k_n} \bigg|\sum_{k=1}^{j} Z_k\bigg| > \delta\bigg].$$

Corollary 2.1 in Peligrad [54] then implies

$$I(u, n) \leqslant \delta^{-2} C \exp\bigg(8 \sum_{j=0}^{\lfloor \log_2 k_n \rfloor} \widetilde{\rho}_{\lfloor 2^{j/3} \rfloor}\bigg)\, k_n \mathrm{E}(Z_1^2),$$

where $(\widetilde{\rho}_k)$ is the ρ-mixing sequence of (Z_k) and C is some positive constant (here we put $\log_2 0 := 0$). Now a little calculations show that for any $k \in \mathbb{N}$,

$$\widetilde{\rho}_k \leqslant \rho_{(k-1)r_n+1},$$

and since the sequence (ρ_k) is non-increasing, we have $\widetilde{\rho}_k \leqslant \rho_k$. From this and assumption (3.50) we obtain that

$$I(u, n) \leqslant CL\delta^{-2}\, k_n \mathrm{E}(Z_1^2), \tag{3.51}$$

for some positive constant L. Further we have

$$\begin{aligned}
\mathrm{E}(Z_1^2) &\leqslant \mathrm{E}\Big(\frac{|S_{r_n}|^2}{a_n^2} 1_{\big\{\frac{|S_{r_n}|}{a_n} \leqslant u\big\}}\Big) = \mathrm{E}\Big(\frac{|S_{r_n}|^2}{a_n^2} 1_{\big\{\frac{|S_{r_n}|}{a_n} \leqslant u\big\}} 1_{\{\cap_{i=1}^{r_n}\{|X_i| \leqslant ua_n\}\}}\Big) \\
&\quad + \mathrm{E}\Big(\frac{|S_{r_n}|^2}{a_n^2} 1_{\big\{\frac{|S_{r_n}|}{a_n} \leqslant u\big\}} 1_{\{\cup_{i=1}^{r_n}\{|X_i| > ua_n\}\}}\Big) \\
&\leqslant \mathrm{E}\Big(\Big|\sum_{i=1}^{r_n} \frac{X_i}{a_n} 1_{\big\{\frac{|X_i|}{a_n} \leqslant u\big\}}\Big|^2\Big) + u^2 \mathrm{P}\Big(\bigcup_{i=1}^{r_n} \{|X_i| > ua_n\}\Big). \qquad (3.52)
\end{aligned}$$

Since the random variables X_i are symmetric, by Theorem 2.1 in Peligrad [54] we have

$$\mathrm{E}\Big(\Big|\sum_{i=1}^{r_n} \frac{X_i}{a_n} 1_{\big\{\frac{|X_i|}{a_n} \leqslant u\big\}}\Big|^2\Big) \leqslant C \exp\Big(8 \sum_{j=0}^{\lfloor \log_2 r_n \rfloor} \rho_{\lfloor 2^{j/3} \rfloor}(n,u)\Big) r_n \mathrm{E}\Big(\frac{X_1^2}{a_n^2} 1_{\big\{\frac{|X_1|}{a_n} \leqslant u\big\}}\Big), \quad (3.53)$$

for all $n \in \mathbb{N}$, where $(\rho_j(n,u))_j$ is the sequence of ρ-mixing coefficients of $\Big(\frac{X_j}{a_n} 1_{\big\{\frac{|X_j|}{a_n} \leqslant u\big\}}\Big)_j$. Since the function $f = f_{n,u} \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{a_n} 1_{\big\{\frac{|x|}{a_n} \leqslant u\big\}}$$

is measurable, it follows that

$$\sigma\Big(\frac{X_j}{a_n} 1_{\big\{\frac{|X_j|}{a_n} \leqslant u\big\}}\Big) \subseteq \sigma(X_j)$$

(see Theorem 4 in Chow and Teicher [20]). From this we immediately obtain $\rho_j(n,u) \leqslant \rho_j$, for all $j, n \in \mathbb{N}$ and $u > 0$. Thus from (3.53), by a new application of assumption (3.50), we get

$$\mathrm{E}\Big(\Big|\sum_{i=1}^{r_n} \frac{X_i}{a_n} 1_{\big\{\frac{|X_i|}{a_n} \leqslant u\big\}}\Big|^2\Big) \leqslant CL\, r_n \mathrm{E}\Big(\frac{X_1^2}{a_n^2} 1_{\big\{\frac{|X_1|}{a_n} \leqslant u\big\}}\Big). \qquad (3.54)$$

Now relations (3.52) and (3.54) imply

$$\begin{aligned}
k_n \mathrm{E}(Z_1^2) &\leqslant CL\, k_n r_n \mathrm{E}\Big(\frac{X_1^2}{a_n^2} 1_{\big\{\frac{|X_1|}{a_n} \leqslant u\big\}}\Big) + u^2 k_n r_n P(|X_1| > ua_n) \\
&= u^2 \cdot \frac{k_n r_n}{n} \cdot n\mathrm{P}(|X_1| > ua_n) \cdot \Big[CL \frac{\mathrm{E}[X_1^2 1_{\{|X_1| \leqslant ua_n\}}]}{u^2 a_n^2 P(|X_1| > ua_n)} + 1\Big].
\end{aligned}$$

From this, using the fact that $k_n r_n/n \to 1$ as $n \to \infty$, regular variation property of X_1 and Theorem 1.12, we obtain

$$\limsup_{n\to\infty} k_n \mathrm{E}(Z_1^2) \leqslant u^{2-\alpha}\Big(\frac{CL\alpha}{2-\alpha}+1\Big).$$

Letting $u \downarrow 0$, it follows that $\lim_{u\downarrow 0}\limsup_{n\to\infty} k_n \mathrm{E}(Z_1^2) = 0$. Therefore, from (3.51), we get

$$\lim_{u\downarrow 0}\limsup_{n\to\infty} I(u,n) = 0,$$

and Condition 3.12 holds. $\square$

Chapter 4

Applications to different time series models

In this chapter we analyze some time series models that are often used in applications. These models include MA, GARCH, ARMA and stochastic volatility models. For the first three of them, we will give sufficient conditions for Theorem 2.15 to hold. Therefore for these models we obtain functional limit theorems with the M_1 convergence. For the stochastic volatility model we are able to obtain a stronger result, since under suitable assumptions, the dependence condition D' will hold. Thus by an application of Theorem 3.7, we will get the J_1 convergence in the corresponding functional limit theorem. At the end of each section we give a proposition which contains the corresponding functional limit theorem for the model in consideration.

4.1 MA models

Consider the finite order MA (moving average) process defined by

$$X_n = \sum_{i=0}^{m} c_i Z_{n-i}, \qquad n \in \mathbb{Z}, \tag{4.1}$$

where $(Z_i)_{i \in \mathbb{Z}}$ is an i.i.d. sequence of regularly varying random variables with index $\alpha \in (0,2)$, $m \in \mathbb{N}$, $c_0, \ldots, c_m$ are nonnegative constants and at least c_0 and c_m are not equal to 0. Take a sequence of positive real numbers (a_n) such that

$$nP(|Z_1| > a_n) \to 1 \qquad \text{as } n \to \infty. \tag{4.2}$$

Clearly the sequence (X_n) is strictly stationary and m-dependent, therefore also strongly mixing, so the mixing condition $\mathcal{A}'(a_n)$ holds by Proposition 1.34.

Let $k \in \mathbb{N}$ be arbitrary. For every $i = 0, 1, \ldots, m$ define matrices $\mathbf{A}_{k,i}$ of dimension $k \times k$ and k–dimensional random vectors $\mathbf{Z}_{k,i}$ by

$$\mathbf{A}_{k,i} = \begin{pmatrix} c_i & 0 & 0 & \ldots & 0 \\ 0 & c_i & 0 & \ldots & 0 \\ 0 & 0 & c_i & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & c_i \end{pmatrix}$$

$$\mathbf{Z}_{k,i} = (Z_{1-i}, \ldots, Z_{k-i})'.$$

Let $\mathbf{X}_{(k)} = (X_1, \ldots, X_k)'$. Then

$$\mathbf{X}_{(k)} = \sum_{i=0}^{m} \mathbf{A}_{k,i}\mathbf{Z}_{k,i}.$$

Since the components of the random vector $\mathbf{Z}_{k,i}$ are i.i.d. and regularly varying with index α, a multidimensional version of Proposition 1.11 implies that the vector $\mathbf{Z}_{k,i}$ is regularly varying with index α. Therefore, by Definition 1.5, there exists a nonzero Radon measure $\mu_{k,i}$ on $(\mathbb{E}^k, \mathcal{B}(\mathbb{E}^k))$ with $\mu_{k,i}(\overline{\mathbb{R}}^k \setminus \mathbb{R}^k) = 0$ such that, as $n \to \infty$,

$$nP\Big(\frac{\mathbf{Z}_{k,i}}{a_n} \in \cdot\Big) \xrightarrow{v} \mu_{k,i}(\cdot).$$

Let $\|\mathbf{x}\|$ denote the Euclidian norm of a vector $\mathbf{x} \in \mathbb{R}^k$, and $\|\mathbf{A}\|$ the operator norm of a $k \times k$– matrix $\mathbf{A}$, i.e. $\|\mathbf{A}\| = \sup_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|$. Since

$$\sum_{i=0}^{m} \|\mathbf{A}_{k,i}\|^{\delta} = \sum_{i=0}^{m} |c_i|^{\delta} < \infty,$$

for every $\delta > 0$, by Theorem 3.1 and Remark 3.2 in Hult and Samorodnitsky [38] we have that, as $n \to \infty$,

$$\frac{\mathrm{P}(a_n^{-1}\mathbf{X}_{(k)} \in \cdot)}{\mathrm{P}(\|\mathbf{Z}_{k,0}\| > a_n)} \xrightarrow{v} \widetilde{\mu_k}(\cdot) := \sum_{i=0}^{m} \mu_{k,i} \circ \mathbf{A}_{k,i}^{-1}(\cdot). \tag{4.3}$$

Since the random variables $Z_1^2, \ldots, Z_k^2$ are i.i.d. and regularly varying (with index $\alpha/2$), we have

$$\mathrm{P}(\|\mathbf{Z}_{k,0}\| > a_n) = \mathrm{P}(Z_1^2 + \ldots + Z_k^2 > a_n^2) \sim k\mathrm{P}(Z_1^2 > a_n^2) = k\mathrm{P}(|Z_1| > a_n)$$

(see Feller [32, p. 271] or Lemma 1.3.1 in Embrechts et al. [30]). Hence from relation (4.2) we obtain that, as $n \to \infty$,

$$n\mathrm{P}(\|\mathbf{Z}_{k,0}\| > a_n) \to k.$$

Therefore from (4.3) we obtain that, as $n \to \infty$,

$$n\mathrm{P}(a_n^{-1}\mathbf{X}_{(k)} \in \cdot) \xrightarrow{v} k\,\widetilde{\mu_k}(\cdot). \tag{4.4}$$

If we show the measure $\widetilde{\mu_k}$ is nonzero, then the random vector $\mathbf{X}_{(k)}$ will be regularly varying with index α. Since the measure $\mu_{k,m}$ is nonzero, there exists a set $B \in \mathcal{B}(\mathbb{E}^k)$ such that $\mu_{k,m}(B) > 0$. But then, by Theorem 1.6 (ii) and the fact that $c_m > 0$, we have

$$\mu_{k,m} \circ \mathbf{A}_{k,m}^{-1}(B) = \mu_{k,m}(\{\mathbf{x} : \mathbf{A}_{k,m}\mathbf{x} \in B\}) = \mu_{k,m}(c_m^{-1}B) = c_m^{\alpha}\,\mu_{k,m}(B) > 0.$$

Therefore $\widetilde{\mu_k}(B) > 0$, and hence for any $k \in \mathbb{N}$ the random vector $\mathbf{X}_{(k)} = (X_1, \ldots, X_k)$ is regularly varying with index α (note that this trivially implies that X_1 is regularly varying with the same index). From this we conclude that the process (X_n) is regularly varying with index α (see Remark 1.14). A different proof of the regular variation property for the process (X_n) appeared earlier in an unpublished work of Basrak and Segers.

In the sequel we assume (without loss of generality) that $\sum_{i=0}^{m} c_i^{\alpha} = 1$. Then from (4.2) and Lemma 1.2 in Cline [21], we have that

$$nP(|X_1| > a_n) \to 1, \qquad \text{as } n \to \infty.$$

Let (r_n) be an arbitrary sequence of positive integers such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$. Choose k and n such that $r_n \geqslant k > m$. Then taking into account stationarity and m-dependency of (X_n) we have, for an arbitrary $t > 0$,

$$\begin{aligned} \mathrm{P}\Big(\max_{k \leqslant |i| \leqslant r_n} |X_i| > ta_n \,\Big|\, |X_0| > ta_n \Big) &= \mathrm{P}\Big(\max_{k \leqslant |i| \leqslant r_n} |X_i| > ta_n \Big) \\ &\leqslant 2(r_n - k + 1)\mathrm{P}(|X_1| > ta_n) \leqslant \frac{2r_n}{n} \cdot n\mathrm{P}(|X_1| > ta_n). \end{aligned}$$

Since X_1 is regularly varying, $n\mathrm{P}(|X_1| > ta_n) \to t^{-\alpha}$ as $n \to \infty$. This with the fact that $r_n/n \to 0$ as $n \to \infty$ yields that

$$\limsup_{n \to \infty} \mathrm{P}\Big(\max_{k \leqslant |i| \leqslant r_n} |X_i| > ta_n \,\Big|\, |X_0| > ta_n \Big) = 0,$$

and therefore the anti-clustering condition $\mathcal{AC}(a_n)$ holds.

The tail process (Y_n) of the process (X_n) can be found by direct calculation. First, recalling the notions and result from Section 1.3, it holds that $Y_0 = |Y_0|\Theta_0$ where $|Y_0|$ and $\Theta_0 = \mathrm{sign}(Y_0)$ are independent with $P(|Y_0| > y) = y^{-\alpha}$ for $y \geqslant 1$ and $\mathrm{P}(\Theta_0 = 1) = p$, $\mathrm{P}(\Theta_0 = -1) = q = 1 - p$. Next, let K denote a random variable with values in the set $\{0, \ldots, m\}$, independent of Y_0, and such that $\mathrm{P}(K = j) = c_j^{\alpha}$. To simplify notation, put $c_i := 0$ for $i \notin \{0, \ldots, m\}$. Then

$$Y_n = \frac{c_{n+K}}{c_K} Y_0, \quad \Theta_n = \frac{c_{n+K}}{c_K} \Theta_0, \qquad n \in \mathbb{Z}, \tag{4.5}$$

represents the tail process and spectral process of (X_n), respectively.

For simplicity, we will prove (4.5) only for $n \in \{1, \ldots, m\}$ and in the case when $c_i > 0$ for all $i = 0, 1, \ldots, m$ (the other cases can be treated similarly, and therefore are omitted). Let $y > 0$ be arbitrary. Firstly, we will calculate

$$\lim_{x\to\infty} \mathrm{P}(x^{-1}X_n > y \,\big|\, |X_0| > x) = \lim_{x\to\infty} \frac{\mathrm{P}(X_n > xy,\, |X_0| > x)}{\mathrm{P}(|X_0| > x)}, \tag{4.6}$$

and show that this expression equals $\mathrm{P}((c_{n+K}/c_K)\, Y_0 > y)$. For every $l \in \{-m, \ldots, n\}$ define the set $A(l) = \{-m, \ldots, n\} \setminus \{l\}$. Then we have

$$\begin{aligned}
\mathrm{P}(X_n > xy,\, |X_0| > x) \;\; &= \;\; \mathrm{P}\Big(\bigcap_{l=-m}^{n} \{X_n > xy,\, |X_0| > x,\, |Z_l| \leqslant x^{2/3}\} \Big) \\
&\quad + \mathrm{P}\Big(\bigcup_{l=-m}^{n} \{X_n > xy,\, |X_0| > x,\, |Z_l| > x^{2/3}, |Z_s| \leqslant x^{2/3} \text{ for all } s \in A(l)\} \Big) \\
&\quad + \mathrm{P}\Big(\bigcup_{-m\leqslant l<s\leqslant n} \{X_n > xy,\, |X_0| > x,\, |Z_l| > x^{2/3}, |Z_s| > x^{2/3}\} \Big) \\
&=: \;\; I_1(x) + I_2(x) + I_3(x).
\end{aligned} \tag{4.7}$$

Since $|Z_{-i}| \leqslant x^{2/3}$ for $i = 0, \ldots, m$ implies

$$|X_0| \leqslant \sum_{i=0}^{m} c_i |Z_{-i}| \leqslant x^{2/3} \sum_{i=0}^{m} c_i \leqslant x, \qquad \text{for large } x,$$

it follows that $I_1(x) = 0$ for large x, which obviously implies

$$\lim_{x\to\infty} \frac{I_1(x)}{\mathrm{P}(|X_0| > x)} = 0. \tag{4.8}$$

The fact that the random variables Z_i are i.i.d. gives

$$I_3(x) \leqslant \sum_{-m\leqslant l<s\leqslant n} \mathrm{P}(|Z_l| > x^{2/3}, |Z_s| > x^{2/3}) = \binom{m+n+1}{2} [\mathrm{P}(|Z_1| > x^{2/3})]^2.$$

Therefore, denoting $M = \binom{m+n+1}{2}$ and using the fact that $\mathrm{P}(|X_0| > x) = x^{-\alpha}L(x)$ for some slowly varying function L (see Proposition 1.8), we have

$$\begin{aligned}
\frac{I_3(x)}{\mathrm{P}(|X_0| > x)} \;\; &\leqslant \;\; M \left[\frac{\mathrm{P}(|Z_1| > x^{2/3})}{\mathrm{P}(|X_0| > x^{2/3})} \right]^2 \cdot \frac{[\mathrm{P}(|X_0| > x^{2/3})]^2}{\mathrm{P}(|X_0| > x)} \\
&= \;\; M \left[\frac{\mathrm{P}(|Z_1| > x^{2/3})}{\mathrm{P}(|X_0| > x^{2/3})} \right]^2 \cdot \frac{[x^{-\alpha/24} L(x^{2/3})]^2}{x^{\alpha/4} L(x)}.
\end{aligned}$$

Since by Lemma 1.2 in Cline [21],

$$\frac{\mathrm{P}(|Z_1| > x^{2/3})}{\mathrm{P}(|X_0| > x^{2/3})} \to \frac{1}{\sum_{i=0}^{m} c_i^{\alpha}} = 1, \qquad \text{as } x \to \infty,$$

and by Proposition 1.3.6 in Bingham et al. [13], $x^{-\alpha/24} L(x^{2/3}) \to 0$ and $x^{\alpha/4} L(x) \to \infty$ as $n \to \infty$, we immediately get

$$\lim_{x\to\infty} \frac{I_3(x)}{\mathrm{P}(|X_0| > x)} = 0. \tag{4.9}$$

The term $I_2(x)$ can be written in the following form

$$\begin{aligned} I_2(x) \;&=\; \sum_{l=-m}^{0} \mathrm{P}(X_n > xy, |X_0| > x, |Z_l| > x^{2/3}, |Z_s| \leqslant x^{2/3} \text{ for all } s \in A(l)) \\ &\quad + \sum_{l=1}^{n} \mathrm{P}(X_n > xy, |X_0| > x, |Z_l| > x^{2/3}, |Z_s| \leqslant x^{2/3} \text{ for all } s \in A(l)) \end{aligned}$$

In a same way as for $I_1(x)$, we obtain that, for every $l = 1, \ldots, n$,

$$\mathrm{P}(X_n > xy, |X_0| > x, |Z_l| > x^{2/3}, |Z_s| \leqslant x^{2/3} \text{ for all } s \in A(l)) = 0$$

for large x. Therefore, for large x,

$$I_2(x) = \sum_{l=-m}^{0} \mathrm{P}(X_n > xy, |X_0| > x, |Z_l| > x^{2/3}, |Z_s| \leqslant x^{2/3} \text{ for all } s \in A(l)). \tag{4.10}$$

Note that for every $l = -m, \ldots, n-m-1$,

$$\mathrm{P}(X_n > xy, |X_0| > x, |Z_l| > x^{2/3}, |Z_s| \leqslant x^{2/3} \text{ for all } s \in A(l)) = 0 \tag{4.11}$$

for large x, since $|Z_s| \leqslant x^{2/3}$ for all $s \in A(l)$ implies

$$|X_n| \leqslant \sum_{i=0}^{m} c_i |Z_{n-i}| \leqslant x^{2/3} \sum_{i=0}^{m} c_i \leqslant xy, \qquad \text{for large } x.$$

Now take an arbitrary $l \in \{n-m, \ldots, 0\}$ and put $B(l) = \{0 \ldots, m\} \setminus \{-l\}$ and $C_l = \sum_{i \in B(l)} c_i = \sum_{i=0}^{m} c_i - c_{-l}$. Then, since $|Z_s| \leqslant x^{2/3}$ for $s \in A(l)$, $|X_0| > x$ and $X_n > xy$ imply

$$x < |X_0| \leqslant c_{-l}|Z_l| + \sum_{i \in B(l)} c_i |Z_{-i}| \leqslant c_{-l}|Z_l| + x^{2/3} C_l,$$

i.e. $c_{-l}|Z_l| > x - x^{2/3}C_l$,

$$xy < |X_n| \leqslant c_{n-l}|Z_l| + \sum_{i \in B(l-n)} c_i|Z_{n-i}| \leqslant c_{n-l}|Z_l| + x^{2/3}C_{l-n},$$

i.e. $c_{n-l}|Z_l| > xy - x^{2/3}C_{l-n}$, and

$$xy < X_n \leqslant c_{n-l}Z_l + x^{2/3}C_{l-n},$$

i.e. $Z_l > (xy - x^{2/3}C_{l-n})/c_{n-l} > 0$ for large x, we have

$$\begin{aligned} &\mathrm{P}(X_n > xy, |X_0| > x, |Z_l| > x^{2/3}, |Z_s| \leqslant x^{2/3} \text{ for all } s \in A(l)) \\ &\leqslant \mathrm{P}(Z_l > 0,\, c_{-l}|Z_l| > x - x^{2/3}C_l,\, c_{n-l}|Z_l| > xy - x^{2/3}C_{l-n}) \end{aligned} \tag{4.12}$$

for large x. Therefore, from (4.10), (4.11) and (4.12) we obtain that, for large x,

$$I_2(x) \leqslant \sum_{l=n-m}^{0} \mathrm{P}(Z_l > 0,\, c_{-l}|Z_l| > x - x^{2/3}C_l,\, c_{n-l}|Z_l| > xy - x^{2/3}C_{l-n}).$$

Now take an arbitrary $s \in (0,1)$. Then for large x it holds that

$$x - x^{2/3}C_l \geqslant sx \qquad \text{and} \qquad xy - x^{2/3}C_{l-n} \geqslant sxy.$$

This implies

$$\begin{aligned} I_2(x) &\leqslant \sum_{l=n-m}^{0} \mathrm{P}(c_{-l}Z_l > sx,\, c_{n-l}Z_l > sxy) \\ &= \sum_{l=n-m}^{0} \mathrm{P}(Z_l > sb_{n,l}(y)x), \qquad \text{for large } x, \end{aligned}$$

where $b_{n,l}(y) = \max\{1/c_{-l},\, y/c_{n-l}\}$. Hence

$$\begin{aligned} \limsup_{x\to\infty} \frac{I_2(x)}{\mathrm{P}(|X_0| > x)} &\leqslant \sum_{l=n-m}^{0} \limsup_{x\to\infty} \frac{\mathrm{P}(Z_l > sb_{n,l}(y)x)}{\mathrm{P}(|X_0| > x)} \\ = \sum_{l=n-m}^{0} \limsup_{x\to\infty} &\frac{\mathrm{P}(Z_l > sb_{n,l}(y)x)}{\mathrm{P}(|Z_l| > sb_{n,l}(y)x)} \cdot \frac{\mathrm{P}(|Z_l| > sb_{n,l}(y)x)}{\mathrm{P}(|X_0| > sb_{n,l}(y)x)} \cdot \frac{\mathrm{P}(|X_0| > sb_{n,l}(y)x)}{\mathrm{P}(|X_0| > x)}. \end{aligned}$$

Since the spectral measure of X_0 is equal to the law of Θ_0 (recall this fact from Section 1.3), we have

$$\lim_{x\to\infty} \frac{\mathrm{P}(X_0 > x)}{\mathrm{P}(|X_0| > x)} = p \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\mathrm{P}(X_0 < -x)}{\mathrm{P}(|X_0| > x)} = q.$$

Put

$$p' = \lim_{x\to\infty} \frac{\mathrm{P}(Z_1 > x)}{\mathrm{P}(|Z_1| > x)} \qquad \text{and} \qquad q' = \lim_{x\to\infty} \frac{\mathrm{P}(Z_1 < -x)}{\mathrm{P}(|Z_1| > x)}.$$

Then by Lemma A3.26 in Embrechts et al. [30] we have

$$\lim_{x\to\infty} \frac{\mathrm{P}(X_0 > x)}{\mathrm{P}(|Z_1| > x)} = p' \qquad \text{and} \qquad \lim_{x\to\infty} \frac{\mathrm{P}(X_0 < -x)}{\mathrm{P}(|Z_1| > x)} = q'.$$

Therefore, using Lemma 1.2 in Cline [21], we obtain that

$$\begin{aligned} p' &= \lim_{x\to\infty} \frac{\mathrm{P}(X_0 > x)}{\mathrm{P}(|Z_1| > x)} = \lim_{x\to\infty} \frac{\mathrm{P}(X_0 > x)}{\mathrm{P}(|X_0| > x)} \cdot \frac{\mathrm{P}(|X_0| > x)}{\mathrm{P}(|Z_1| > x)} \\ &= p \sum_{i=0}^{m} c_i^{\alpha} = p. \end{aligned}$$

Similarly $q' = q$. This implies

$$\lim_{x\to\infty} \frac{\mathrm{P}(Z_l > sb_{n,l}(y)x)}{\mathrm{P}(|Z_l| > sb_{n,l}(y)x)} = p.$$

Using again Lemma 1.2 in [21] we get

$$\lim_{x\to\infty} \frac{\mathrm{P}(|Z_l| > sb_{n,l}(y)x)}{\mathrm{P}(|X_0| > sb_{n,l}(y)x)} = 1.$$

Further, since X_0 is regularly varying with index α, it follows that

$$\lim_{x\to\infty} \frac{\mathrm{P}(|X_0| > sb_{n,l}(y)x)}{\mathrm{P}(|X_0| > x)} = (sb_{n,l}(y))^{-\alpha}.$$

This all leads to

$$\limsup_{x\to\infty} \frac{I_2(x)}{\mathrm{P}(|X_0| > x)} \leqslant ps^{-\alpha} \sum_{l=n-m}^{0} (b_{n,l}(y))^{-\alpha}.$$

Since $s \in (0,1)$ was arbitrary, letting $s \to 1$, we obtain that

$$\limsup_{x\to\infty} \frac{I_2(x)}{\mathrm{P}(|X_0| > x)} \leqslant p \sum_{l=n-m}^{0} (b_{n,l}(y))^{-\alpha}. \tag{4.13}$$

On the other hand, for an arbitrary $l \in \{n-m, \dots, 0\}$, since $c_{-l}Z_l > x + x^{2/3}C_l$, $c_{n-l}Z_l > xy + x^{2/3}C_{l-n}$ and $|Z_s| \leqslant x^{2/3}$ for $s \in A(l)$ imply

$$X_n = c_{n-l}Z_l + \sum_{i\in B(l-n)} c_i Z_{n-i} > xy + x^{2/3}C_{l-n} - x^{2/3}C_{l-n} = xy,$$

$$Z_l > (xy + x^{2/3}C_{l-n})/c_{n-l} > x^{2/3}, \qquad \text{for large } x,$$

and

$$x + x^{2/3}C_l < c_{-l}|Z_l| = \Big|X_0 - \sum_{i\in B(l)} c_i Z_{-i}\Big| \leqslant |X_0| + \sum_{i\in B(l)} c_i |Z_{-i}| \leqslant |X_0| + x^{2/3}C_l,$$

i.e. $|X_0| > x$, we obtain that

$$\begin{aligned} &\mathrm{P}(c_{-l}Z_l > x + x^{2/3}C_l, c_{n-l}Z_l > xy + x^{2/3}C_{l-n}, |Z_s| \leqslant x^{2/3} \,\text{for all}\, s \in A(l)) \\ &\leqslant \mathrm{P}(X_n > xy, |X_0| > x, |Z_l| > x^{2/3}, |Z_s| \leqslant x^{2/3} \,\text{for all}\, s \in A(l)) \end{aligned} \tag{4.14}$$

for large x. Therefore, from (4.10), (4.11) and (4.14) we obtain that, for large x,

$$I_2(x) \geqslant \sum_{l=n-m}^{0} \mathrm{P}\big[c_{-l}Z_l > x + x^{2/3}C_l, c_{n-l}Z_l > xy + x^{2/3}C_{l-n}, |Z_s| \leqslant x^{2/3} \text{ for all}\, s \in A(l)\big].$$

Now take an arbitrary $s \in (1,2)$. Then for large x it holds that

$$x + x^{2/3}C_l \leqslant sx \qquad \text{and} \qquad xy + x^{2/3}C_{l-n} \leqslant sxy.$$

This implies

$$\begin{aligned} I_2(x) &\geqslant \sum_{l=n-m}^{0} \mathrm{P}(c_{-l}Z_l > sx,\, c_{n-l}Z_l > sxy,\, |Z_s| \leqslant x^{2/3} \,\text{for all}\, s \in A(l)) \\ &= \sum_{l=n-m}^{0} \mathrm{P}(Z_l > sb_{n,l}(y)x,\, |Z_s| \leqslant x^{2/3} \,\text{for all}\, s \in A(l)), \qquad \text{for large } x. \end{aligned}$$

Hence, since the random variables Z_i are i.i.d., we obtain that

$$\begin{aligned}\liminf_{x\to\infty}\frac{I_2(x)}{\mathrm{P}(|X_0|>x)} &\geqslant \sum_{l=n-m}^{0}\liminf_{x\to\infty}\frac{\mathrm{P}(Z_l>sb_{n,l}(y)x,\,|Z_s|\leqslant x^{2/3}\,\text{for all}\,s\in A(l))}{\mathrm{P}(|X_0|>x)}\\ &= \sum_{l=n-m}^{0}\liminf_{x\to\infty}\frac{\mathrm{P}(Z_l>sb_{n,l}(y)x)}{\mathrm{P}(|X_0|>x)}\cdot[\mathrm{P}(|Z_1|\leqslant x^{2/3})]^{n+m}\end{aligned}$$

From the calculations that we have already made, it is clear that

$$\lim_{x\to\infty}\frac{\mathrm{P}(Z_l>sb_{n,l}(y)x)}{\mathrm{P}(|X_0|>x)}=p\,(sb_{n,l}(y))^{-\alpha}.$$

This and the fact that $\mathrm{P}(|Z_1|\leqslant x^{2/3})\to 1$ as $x\to\infty$, imply

$$\liminf_{x\to\infty}\frac{I_2(x)}{\mathrm{P}(|X_0|>x)}\geqslant ps^{-\alpha}\sum_{l=n-m}^{0}(b_{n,l}(y))^{-\alpha}.$$

Since $s\in(1,2)$ was arbitrary, letting $s\to 1$, we obtain that

$$\liminf_{x\to\infty}\frac{I_2(x)}{\mathrm{P}(|X_0|>x)}\geqslant p\sum_{l=n-m}^{0}(b_{n,l}(y))^{-\alpha}. \tag{4.15}$$

From relations (4.13) and (4.15) we conclude that

$$\lim_{x\to\infty}\frac{I_2(x)}{\mathrm{P}(|X_0|>x)}= p\sum_{l=n-m}^{0}(b_{n,l}(y))^{-\alpha}. \tag{4.16}$$

Now, relations (4.6), (4.7), (4.8), (4.9) and (4.16) give

$$\lim_{x\to\infty}\mathrm{P}(x^{-1}X_n>y\,\big|\,|X_0|>x)= p\sum_{l=n-m}^{0}(b_{n,l}(y))^{-\alpha}. \tag{4.17}$$

From the independency of Y_0 and K, as well as $|Y_0|$ and Θ_0, and the fact that $c_i=0$ for $i>m$, we get

$$\begin{aligned}\mathrm{P}\Big(\frac{c_{n+K}}{c_K}Y_0>y\Big) &= \sum_{j=0}^{m}\mathrm{P}\Big(\frac{c_{n+K}}{c_K}Y_0>y,\,K=j\Big)=\sum_{j=0}^{m-n}\mathrm{P}\Big(\frac{c_{n+j}}{c_j}Y_0>y\Big)\mathrm{P}(K=j)\\ &= \sum_{j=0}^{m-n}\mathrm{P}\Big(\frac{c_{n+j}}{c_j}|Y_0|\,\Theta_0>y,\,\Theta_0=1\Big)\,c_j^{\alpha}=\sum_{j=0}^{m-n}\mathrm{P}\Big(\frac{c_{n+j}}{c_j}|Y_0|>y\Big)\mathrm{P}(\Theta_0=1)\,c_j^{\alpha}\\ &= p\sum_{j=0}^{m-n}\mathrm{P}\Big(|Y_0|>\frac{yc_j}{c_{n+j}}\Big)\,c_j^{\alpha}.\end{aligned}$$

Since for $z > 0$, $\mathrm{P}(|Y_0| > z) = 1_{[0,1)}(z) + z^{-\alpha} 1_{[1,\infty)}(z)$, after some standard calculations, we obtain that

$$\mathrm{P}\Big(|Y_0| > \frac{y c_j}{c_{n+j}}\Big)\, c_j^{\alpha} = (b_{n,-j}(y))^{-\alpha}.$$

Therefore

$$\mathrm{P}\Big(\frac{c_{n+K}}{c_K}\, Y_0 > y\Big) = p \sum_{j=0}^{m-n} (b_{n,-j}(y))^{-\alpha}. \tag{4.18}$$

Comparing (4.17) and (4.18) we conclude that

$$\lim_{x\to\infty} \mathrm{P}(x^{-1} X_n > y \,\big|\, |X_0| > x) = \mathrm{P}\Big(\frac{c_{n+K}}{c_K}\, Y_0 > y\Big), \qquad y > 0. \tag{4.19}$$

In a similar manner we get

$$\lim_{x\to\infty} \mathrm{P}(x^{-1} X_n \leqslant -y \,\big|\, |X_0| > x) = \mathrm{P}\Big(\frac{c_{n+K}}{c_K}\, Y_0 \leqslant -y\Big), \qquad y > 0. \tag{4.20}$$

Recalling the definition of the tail process in Section 1.3, relations (4.19) and (4.20) yield the first equation in (4.5). The second one then follows immediately. From (4.5) we conclude that at most $m+1$ values Y_n and Θ_n are different from 0 and all have the same sign.

Observe further that, since the sequence (X_n) is m–dependent, it is also ρ–mixing ($\rho_n = 0$ for $n \geqslant m+1$). Hence by Proposition 2.19, Condition 2.14 holds when $\alpha \in [1,2)$. Therefore, the sequence (X_n) satisfies all the conditions of Theorem 2.15, and the partial sum process $V_n(\,\cdot\,)$, defined by (2.4), converges in distribution, as $n \to \infty$, to an α–stable Lévy process $V(\,\cdot\,)$ in $D[0,1]$ under the M_1 topology. The characteristic triple of the limiting process can be found from the results in Davis and Resnick [23] and Davis and Hsing [24]. Suppose that $\alpha \in (0,1) \cup (1,2)$ and $\mathrm{E}Z_1 = 0$ for $\alpha > 1$. Then the characteristic triple is of the form $(0, \nu, b)$, where

$$\nu(dx) = \alpha \Big(\sum_{i=0}^{m} c_i\Big)^{\alpha} |x|^{-1-\alpha} \Big(p 1_{(0,\infty)}(x) + q 1_{(-\infty,0)}(x)\Big)\, dx, \tag{4.21}$$

$$b = (p - q)\frac{\alpha}{1-\alpha}\Big[\Big(\sum_{i=0}^{m} c_i\Big)^{\alpha} - 1\Big]. \tag{4.22}$$

The case when $\alpha = 1$ can be treated similarly, but the corresponding expressions are much more complicated, due to the specific form of the location parameter τ of the characteristic function of $V(1)$ represented in the form given in Theorem 1.45 (see Remark 3.2 and Theorem 3.2 in Davis and Hsing [24]).

The following proposition concisely gives the functional limit theorem for finite order MA processes considered in this section.

Proposition 4.1. *Let (X_n) be a finite order MA process defined by (4.1), where (Z_i) is an i.i.d. sequence of regularly varying random variables with index $\alpha \in (0,2)$, the coefficients $c_0, c_1, \ldots, c_m$ are nonnegative and at least c_0 and c_m are not equal to zero. Then the following statements hold.*

1. *The partial sum stochastic process $V_n(\,\cdot\,)$, defined by (2.4), converges in distribution to an α–stable Lévy process $V(\,\cdot\,)$ in $D[0,1]$ under the M_1 topology.*

2. *Assume $\alpha \in (0,1) \cup (1,2)$, $\sum_{i=0}^{m} c_i^{\alpha} = 1$ and $\mathrm{E}Z_1 = 0$ for $\alpha > 1$. Then the characteristic triple of the limiting process $V(\,\cdot\,)$ is of the form $(0, \nu, b)$, where ν and b are given in (4.21) and (4.22).*

Infinite order MA processes with nonnegative coefficients are considered in Avram and Taqqu [3]. In principle, one can approximate such processes by a sequence of finite order MA processes, for which Theorem 2.15 applies, and show that the error of approximation is negligible in the limit. We decided not to pursue this here, since the functional limit theorem for these processes already appears in [3].

4.2 ARCH/GARCH models

We consider the model

$$X_n = \sigma_n Z_n, \tag{4.23}$$

where $(Z_n)_{n\in\mathbb{Z}}$ is a sequence of i.i.d. random variables with $\mathrm{E}Z_1 = 0$ and $\mathrm{Var}\, Z_1 = 1$, and

$$\sigma_n^2 = \alpha_0 + (\alpha_1 Z_{n-1}^2 + \beta_1)\sigma_{n-1}^2. \tag{4.24}$$

Assume that $\alpha_0 > 0$ and the non-negative parameters α_1, β_1 are chosen such that a strictly stationary solution to the equation (4.24) exists, namely

$$-\infty \leqslant \mathrm{E}\ln(\alpha_1 Z_1^2 + \beta_1) < 0$$

(see Goldie [34] and Mikosch and Stărică [51]). The process (X_n) is then strictly stationary too. If $\alpha_1 > 0$ and $\beta_1 > 0$ it is called a GARCH(1,1) process, while if $\alpha_1 > 0$ and $\beta_1 = 0$ it is called an ARCH(1) process.

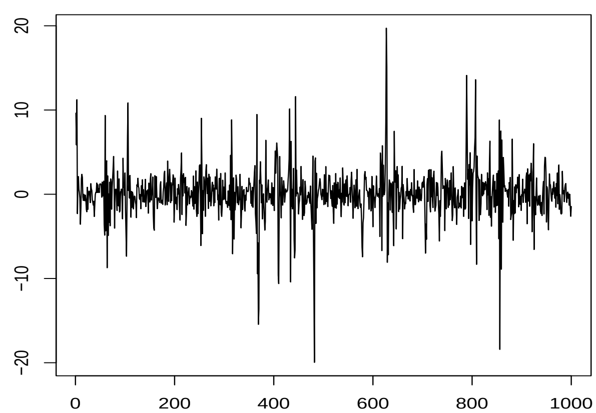

Figure 4.1: *A simulated GARCH(1,1) process with* $\alpha_0 = 1$, $\alpha_1 = 0.7$ *and* $\beta_1 = 0.2$. *The noise* (Z_n) *is i.i.d. standard normal.*

In the rest of the section we consider a stationary squared GARCH(1,1) process (X_n^2). Assume that Z_1 is symmetric, has a positive Lebesgue density on $\mathbb{R}$ and there exists $\alpha \in (0,2)$ such that

$$\mathrm{E}[(\alpha_1 Z_1^2 + \beta_1)^\alpha] = 1 \quad \text{and} \quad \mathrm{E}[(\alpha_1 Z_1^2 + \beta_1)^\alpha \ln(\alpha_1 Z_1^2 + \beta_1)] < \infty.$$

Then it is known that the processes (σ_n^2) and (X_n^2) are regularly varying with index α and strongly mixing with geometric rate (see Bartkiewicz et al. [6], Basrak et al. [9] and Mikosch and Stărică [51]). Proposition 3 in Breiman [17][1] and Proposition 4.7 in Bartkiewicz et al. [6] imply

$$\mathrm{P}(X_1^2 > x) \sim \mathrm{E}|Z_1|^{2\alpha}\mathrm{P}(\sigma_1^2 > x) \sim \mathrm{E}|Z_1|^{2\alpha} c_1 x^{-\alpha}, \qquad \text{as } x \to \infty, \tag{4.25}$$

where

$$c_1 = \frac{\mathrm{E}[(\alpha_0 + (\alpha_1 Z_1^2 + \beta_1)\sigma_1^2)^\alpha - (\alpha_1 Z_1^2 + \beta_1)^\alpha]}{\alpha \mathrm{E}[(\alpha_1 Z_1^2 + \beta_1)^\alpha \ln(\alpha_1 Z_1^2 + \beta_1)]} > 0.$$

The squared GARCH(1,1) process can be embedded in the 2-dimensional stochastic recurrence equation (SRE):

$$\mathbf{X}_n = \mathbf{A}_n \mathbf{X}_{n-1} + \mathbf{B}_n. \tag{4.26}$$

To see this, write

$$\mathbf{X}_n = \begin{pmatrix} X_n^2 \\ \sigma_n^2 \end{pmatrix}, \quad \mathbf{A}_n = \begin{pmatrix} \alpha_1 Z_n^2 & \beta_1 Z_n^2 \\ \alpha_1 & \beta_1 \end{pmatrix}, \quad \mathbf{B}_n = \begin{pmatrix} \alpha_0 Z_n^2 \\ \alpha_0 \end{pmatrix}.$$

Then $(\mathbf{X}_n)$ satisfies the SRE in (4.26). Stochastic recurrence equations have been largely studied, see for example Babillot et al. [4], Basrak et al. [9], Bougerol and Picard [14], Goldie [34], Mikosch and Stărică [51]. Write $Y = \max\{Z_1^2, 1\}$. Then $\|\mathbf{X}_1\| = Y\sigma_1^2$, where $\|\cdot\|$ is the "sup" norm on $\mathbb{R}^2$. Since Y and σ_1^2 are independent nonnegative random variables such that $\mathrm{E}Y < \infty$ and σ_1^2 is regularly varying with index α, Proposition 3 in Breiman [17] implies

$$\mathrm{P}(\|\mathbf{X}_1\| > x) = P(Y\sigma_1^2 > x) \sim \mathrm{E}Y^\alpha \mathrm{P}(\sigma_1^2 > x) \sim \mathrm{E}Y^\alpha c_1 x^{-\alpha} \qquad \text{as } x \to \infty, \tag{4.27}$$

[1]Breiman did not prove this result for general α, but only for $\alpha \in (0,1)$. However, the proof works for all $\alpha > 0$.

i.e. the random variable $\|\mathbf{X}_1\|$ is regularly varying with index α.

Assume

$$\mathrm{E}(\ln \|\mathbf{A}_1\|) < 0, \tag{4.28}$$

where for a 2×2-matrix $\mathbf{A}$,

$$\|\mathbf{A}\| = \sup_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|$$

is the operator norm. One sufficient condition for relation (4.28) to hold is that $\alpha_1 + \beta_1 < 1$. Since $\mathrm{E}\|\mathbf{A}_1\| < \infty$, from Remark 2.9 in [9] it follows

$$\mathrm{E}(\|\mathbf{A}_1\|^{\epsilon}) < 1, \qquad \text{for some } \epsilon \in (0, \alpha \wedge 1]. \tag{4.29}$$

In order to show that the sequence (X_n^2) satisfies the anti-clustering condition $\mathcal{AC}(a_n)$, where (a_n) is a sequence of positive real numbers such that $n\mathrm{P}(X_1^2 > a_n) \to 1$ as $n \to \infty$ (from (4.25) it follows that we can take $a_n = (c_1 \mathrm{E}|Z_1|^{2\alpha} n)^{1/\alpha})$, we first show that this condition is satisfied by the sequence $(\mathbf{X}_n)$, and for this we will use a technique used in the proof of Theorem 2.10 in [9]. Choose some $s \in (0, \epsilon/\alpha)$ and define $r_n = \lfloor n^s \rfloor$. Then

$$\frac{r_n}{a_n^{\epsilon}} \to 0, \qquad \text{as } n \to \infty. \tag{4.30}$$

Iterating (4.26) we obtain

$$\mathbf{X}_i = \prod_{j=1}^{i} \mathbf{A}_j \, \mathbf{X}_0 + \sum_{j=1}^{i} \prod_{m=j+1}^{i} \mathbf{A}_m \, \mathbf{B}_j =: I_{i,1} \, \mathbf{X}_0 + I_{i,2}, \qquad i \in \mathbb{N}.$$

This and the fact that $\mathbf{X}_0$ and $I_{i,2}$ are independent, imply, for an arbitrary $u > 0$,

$$\begin{aligned} &\mathrm{P}(\|\mathbf{X}_i\| > ua_n \,\big|\, \|\mathbf{X}_0\| > ua_n) \\ &\quad \leqslant \mathrm{P}(\|\mathbf{X}_0\| \, \|I_{i,1}\| > ua_n/2 \,\big|\, \|\mathbf{X}_0\| > ua_n) + \mathrm{P}(\|I_{i,2}\| > ua_n/2). \end{aligned} \tag{4.31}$$

Then using Markov's inequality and Theorem 1.12 (note that $\|\mathbf{X}_0\|$ is regularly varying with index α), the limes superior of the first term on the right-hand side of (4.31) is bounded above by

$$\limsup_{n\to\infty} \mathrm{E}(\|I_{i,1}\|^{\epsilon}) 2^{\epsilon} \frac{\mathrm{E}[\|\mathbf{X}_0\|^{\epsilon} 1_{\{\|\mathbf{X}_0\| > ua_n\}}]}{(ua_n)^{\epsilon} \mathrm{P}(\|\mathbf{X}_0\| > ua_n)} \leqslant 2^{\epsilon} \frac{\alpha}{\alpha - \epsilon} (\mathrm{E}\|\mathbf{A}_1\|^{\epsilon})^{i}. \tag{4.32}$$

Further, it holds that

$$\|I_{i,2}\| \stackrel{d}{=} \Big\| \sum_{j=1}^{i} \prod_{m=1}^{j-1} \mathbf{A}_m \, \mathbf{B}_j \Big\| \leqslant \sum_{j=1}^{i} \prod_{m=1}^{j-1} \|\mathbf{A}_m\| \, \|\mathbf{B}_j\|.$$

Using Markov's inequality, the fact that, since $\epsilon \in (0,1)$, $(x_1 + \ldots + x_n)^{\epsilon} \leqslant x_1^{\epsilon} + \ldots + x_n^{\epsilon}$ for all $n \in \mathbb{N}$ and nonnegative $x_1, \ldots, x_n$, and the i.i.d. property of the sequence (Z_i), we get

$$\begin{aligned} \mathrm{P}(\|I_{i,2}\| > ua_n/2) &\leqslant a_n^{-\epsilon} (2/u)^{\epsilon} \, \mathrm{E}\|\mathbf{B}_1\|^{\epsilon} \sum_{j=1}^{i} (\mathrm{E}\|\mathbf{A}_1\|^{\epsilon})^{j-1} \\ &\leqslant a_n^{-\epsilon} (2/u)^{\epsilon} \, \mathrm{E}\|\mathbf{B}_1\|^{\epsilon} \sum_{j=0}^{\infty} (\mathrm{E}\|\mathbf{A}_1\|^{\epsilon})^{j}. \\ &= C a_n^{-\epsilon}, \end{aligned} \tag{4.33}$$

where $C = (2/u)^{\epsilon} \, \mathrm{E}\|\mathbf{B}_1\|^{\epsilon} \sum_{j=0}^{\infty} (\mathrm{E}\|\mathbf{A}_1\|^{\epsilon})^{j} < \infty$ by (4.29). Since from our assumptions it follows that the sequence $(\mathbf{X}_n)$ is strictly stationary (see for example Basrak et al. [9] and Kesten [41]), we have

$$\mathrm{P}(\|\mathbf{X}_{-i}\| > ua_n \,\big|\, \|\mathbf{X}_0\| > ua_n) = \mathrm{P}(\|\mathbf{X}_i\| > ua_n \,\big|\, \|\mathbf{X}_0\| > ua_n).$$

This and relations (4.30), (4.31), (4.32) and (4.33), then imply

$$\begin{aligned} &\lim_{m\to\infty} \limsup_{n\to\infty} \mathrm{P}\Big(\max_{m \leqslant |i| \leqslant r_n} \|\mathbf{X}_i\| > ua_n \,\Big|\, \|\mathbf{X}_0\| > ua_n \Big) \\ &\quad \leqslant \lim_{m\to\infty} \limsup_{n\to\infty} \sum_{m \leqslant |i| \leqslant r_n} \mathrm{P}(\|\mathbf{X}_i\| > ua_n \,\big|\, \|\mathbf{X}_0\| > ua_n) \\ &\quad \leqslant \lim_{m\to\infty} 2^{\epsilon+1} \frac{\alpha}{\alpha - \epsilon} \sum_{i=m}^{\infty} (\mathrm{E}\|\mathbf{A}_1\|^{\epsilon})^{i} \\ &\quad = 0, \end{aligned} \tag{4.34}$$

where the last equation follows from (4.29). Hence condition $\mathcal{AC}(a_n)$ for the sequence $(\mathbf{X}_n)$ holds. Then from (4.25), (4.27) and the fact that $\|\mathbf{X}_i\| \geqslant X_i^2$, we get

$$\begin{aligned}
&\lim_{m\to\infty}\limsup_{n\to\infty} \mathrm{P}\Big(\max_{m\leqslant |i|\leqslant r_n} X_i^2 > ua_n \,\Big|\, X_0^2 > ua_n \Big) \\
&\quad\leqslant \lim_{m\to\infty}\limsup_{n\to\infty} \mathrm{P}\Big(\max_{m\leqslant |i|\leqslant r_n} \|\mathbf{X}_i\| > ua_n \,\Big|\, \|\mathbf{X}_0\| > ua_n \Big) \cdot \frac{\mathrm{P}(\|\mathbf{X}_0\| > ua_n)}{\mathrm{P}(X_0^2 > ua_n)} \\
&\quad= 0 \cdot \frac{\mathrm{E}Y^{\alpha}}{\mathrm{E}|Z_1|^{2\alpha}} = 0.
\end{aligned}$$

Therefore the sequence (X_n^2) satisfies condition $\mathcal{AC}(a_n)$.

Define the sequence (l_n) by $l_n = \lfloor n^{s/2} \rfloor$. If (α_n) denotes the sequence of α–mixing coefficients of (X_n^2), then since α_n converges to zero geometrically fast, standard calculations give, as $n \to \infty$,

$$k_n \alpha_{l_n+1} \to 0 \qquad \text{and} \qquad \frac{k_n l_n}{n} \to 0,$$

where $k_n = \lfloor n/r_n \rfloor$. This corresponds to relation (1.17) in the proof of Proposition 1.34. Following the line of the argument in the proof of that proposition, we conclude that the mixing condition $\mathcal{A}'(a_n)$ holds.

Since the process (X_n^2) is nonnegative, its tail process cannot have two values of the opposite sign. If additionally Condition 2.14 holds when $\alpha \in [1,2)$, then by Theorem 2.15, the partial sum stochastic process $V_n(\,\cdot\,)$, defined by

$$V_n(t) = \sum_{k=1}^{[nt]} \frac{X_k^2}{a_n} - \lfloor nt \rfloor \mathrm{E}\Big(\frac{X_1^2}{a_n} 1_{\left\{ \frac{X_1^2}{a_n} \leqslant 1 \right\}} \Big), \qquad t \in [0,1], \tag{4.35}$$

converges in distribution to an α–stable Lévy process $V(\,\cdot\,)$ in $D[0,1]$ under the M_1 topology. Here $(a_n)_n$ is a positive sequence such that $n\mathrm{P}(X_0^2 > a_n) \to 1$ as $n \to \infty$.

In case $\alpha \in (0,1) \cup (1,2)$, the characteristic triple $(0, \nu, b)$ of the stable random variable $V(1)$ and thus of the stable Lévy process $V(\,\cdot\,)$ can be determined from Bartkiewicz et al. [6, Proposition 4.8], Davis and Hsing [24, Remark 3.1] and Remark 2.17: after some calculations, we find

$$\nu(dx) = c_+ \, 1_{(0,\infty)}(x) \, \alpha x^{-\alpha-1} \, dx, \qquad\qquad b = \frac{\alpha}{1-\alpha}(c_+ - 1), \tag{4.36}$$

where

$$c_+ = \frac{\mathrm{E}[(Z_0^2 + \widetilde{T}_\infty)^\alpha - \widetilde{T}_\infty^\alpha]}{\mathrm{E}(|Z_1|^{2\alpha})}, \qquad\qquad \widetilde{T}_\infty = \sum_{t=1}^{\infty} Z_{t+1}^2 \prod_{i=1}^{t} (\alpha_1 Z_i^2 + \beta_1). \tag{4.37}$$

Therefore, the functional limit result for the squares of GARCH(1,1) process is given in the following proposition.

Proposition 4.2. *Let (X_n) be a strictly stationary GARCH(1,1) process defined by (4.23) and (4.24), where (Z_i) is an i.i.d. sequence of random variables with $\mathrm{E}Z_1 = 0$ and $\mathrm{Var}\, Z_1 = 1$, and the coefficients α_0, α_1 and β_1 are positive. Assume Z_1 is symmetric, has a positive Lebesgue density on $\mathbb{R}$, and*

$$\mathrm{E}[(\alpha_1 Z_1^2 + \beta_1)^\alpha] = 1, \qquad \mathrm{E}[(\alpha_1 Z_1^2 + \beta_1)^\alpha \ln(\alpha_1 Z_1^2 + \beta_1)] < \infty,$$

for some $\alpha \in (0, 2)$. Suppose further that $\mathrm{E}(\ln \|\mathbf{A}_1\|) < 0$ and Condition 2.14 holds when $\alpha \in [1, 2)$. Then the following statements hold.

1. *The partial sum process $V_n(\,\cdot\,)$, defined by (4.35), converges in distribution to an α–stable Lévy process $V(\,\cdot\,)$ in $D[0,1]$ under the M_1 topology.*

2. *Assume $\alpha \in (0,1) \cup (1,2)$. Then the characteristic triple $(0, \nu, b)$ of the limiting process $V(\,\cdot\,)$ is given by (4.36) and (4.37).*

4.3 ARMA models

Suppose a strictly stationary sequence $(X_n)_{n\in\mathbb{Z}}$ satisfies the ARMA(p,q) recursions

$$X_n = \phi_1 X_{n-1} + \cdots + \phi_p X_{n-p} + Z_n + \theta_1 Z_{n-1} + \cdots + \theta_q Z_{n-q}, \tag{4.38}$$

where the coefficients $\phi_1, \ldots, \phi_p, \theta_1, \ldots \theta_q$ are positive, $\Phi(z) := 1 - \phi_1 z - \cdots - \phi_p z^p \neq 0$ for all $|z| \leqslant 1$, and (Z_n) is an i.i.d. sequence of random variables. Assume Z_1 regularly

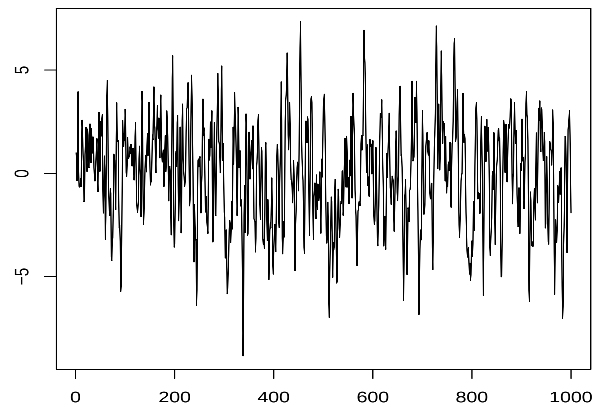

Figure 4.2: *A simulated ARMA(1,1) process with $\phi_1 = 0.6$ and $\theta_1 = 1.2$. The noise (Z_n) is i.i.d. standard normal.*

varying with index $\alpha \in (0, 2)$ and $\mathrm{E}Z_1 = 0$ if $\alpha \in (1, 2)$. By results in Brockwell and Davis [18], (X_n) has the causal representation

$$X_n = \sum_{j=0}^{\infty} \psi_j Z_{n-j}, \tag{4.39}$$

where the coefficients ψ_j can be found from the relation

$$\Phi(z) \sum_{j=0}^{\infty} \psi_j z^j = 1 + \theta_1 z + \cdots + \theta_q z^q. \tag{4.40}$$

Assume the sequence (ψ_j) satisfies the following condition

$$\sum_{j=0}^{\infty} |\psi_j|^\delta < \infty, \qquad \text{for some } 0 < \delta < \min\{1, \alpha\}. \tag{4.41}$$

Then it follows that X_1 is regularly varying with index α,

$$\lim_{x \to \infty} \frac{\mathrm{P}(X_1 > x)}{\mathrm{P}(|X_1| > x)} = \frac{\sum_{j=0}^{\infty} |\psi_j|^\alpha [p\, 1_{(0,\infty)}(\psi_j) + q\, 1_{(-\infty,0)}(\psi_j)]}{\sum_{j=0}^{\infty} |\psi_j|^\alpha}$$

and

$$\lim_{x\to\infty} \frac{\mathrm{P}(X_1 < -x)}{\mathrm{P}(|X_1| > x)} = \frac{\sum_{j=0}^{\infty} |\psi_j|^\alpha [q\, 1_{(0,\infty)}(\psi_j) + p\, 1_{(-\infty,0)}(\psi_j)]}{\sum_{j=0}^{\infty} |\psi_j|^\alpha},$$

where

$$p = \lim_{x\to\infty} \frac{\mathrm{P}(Z_1 > x)}{\mathrm{P}(|Z_1| > x)} \quad \text{and} \quad q = \lim_{x\to\infty} \frac{\mathrm{P}(Z_1 < -x)}{\mathrm{P}(|Z_1| > x)}$$

(see for instance Lemma A.3 in Mikosch and Samorodnitsky [50]; cf. Cline [21] and Kokoszka and Taqqu [42]).

Let $n \in \mathbb{N}$ be arbitrary. For every $j \geqslant 0$ define matrices $\mathbf{A}_{n,j}$ of dimension $n \times n$ and n–dimensional random vectors $\mathbf{Z}_{n,j}$ by

$$\mathbf{A}_{n,j} = \begin{pmatrix} \psi_j & 0 & 0 & \dots & 0 \\ 0 & \psi_j & 0 & \dots & 0 \\ 0 & 0 & \psi_j & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \psi_j \end{pmatrix}$$

$$\mathbf{Z}_{n,j} = (Z_{1-j}, Z_{2-j}, \dots, Z_{n-j})'.$$

Then $\mathbf{Z}_{n,j}$ is regularly varying with index α and

$$\sum_{j=0}^{\infty} \|\mathbf{A}_{n,j}\|^\delta = \sum_{j=0}^{\infty} |\psi_j|^\delta < \infty.$$

Now following the line of the argument given in Example 4.1, we conclude that the random vector

$$(X_1, \dots, X_n) = \sum_{j=0}^{\infty} \mathbf{A}_{n,j} \mathbf{Z}_{n,j}$$

is regularly varying with index α. Therefore, the process (X_n) is regularly varying with index α.

For simplicity we restrict our attention to the ARMA(1,1) model. Note that $\Phi(z) \neq 0$ for all $|z| \leqslant 1$ implies $\phi_1 < 1$. From (4.40) we find that $\psi_0 = 1$ and $\psi_j = (\phi_1 + \theta_1)\, \phi_1^{j-1}$ for all $j \in \mathbb{N}$. Therefore condition (4.41) holds. From the recursions in (4.38) and the representation in (4.39) we have, for any $i \in \mathbb{N}$,

$$\begin{aligned} X_i &= \phi_1^i X_0 + Z_i + (\phi_1 + \theta_1)\sum_{k=0}^{i-2} \phi_1^k Z_{i-k-1} + \phi_1^{i-1}\theta_1 Z_0 \\ &= \phi_1^i X_0 + \sum_{k=0}^{i-1} \psi_k Z_{i-k} + \phi_1^{i-1}\theta_1 Z_0. \qquad (4.42) \end{aligned}$$

To check the anti-clustering condition $\mathcal{AC}(a_n)$ we proceed in the following way. Since X_0 and $(Z_k)_{k\geqslant 1}$ are independent and the sequence $(Z_k)_{k\in\mathbb{Z}}$ is strictly stationary, we have

$$\begin{aligned} & \mathrm{P}\Big(\max_{m\leqslant i\leqslant r_n} |X_i| > ta_n \,\Big|\, |X_0| > ta_n\Big) \\ \leqslant\; & \mathrm{P}\Big(\max_{m\leqslant i\leqslant r_n} \phi_1^i |X_0| > \frac{ta_n}{3} \,\Big|\, |X_0| > ta_n\Big) + \mathrm{P}\Big(\max_{m\leqslant i\leqslant r_n} \Big|\sum_{k=0}^{i-1}\psi_k Z_{i-k}\Big| > \frac{ta_n}{3}\Big) \\ & + \mathrm{P}\Big(\max_{m\leqslant i\leqslant r_n} \phi_1^{i-1}\theta_1 |Z_0| > \frac{ta_n}{3} \,\Big|\, |X_0| > ta_n\Big) \\ \leqslant\; & \mathrm{P}\Big(\phi_1^m |X_0| > \frac{ta_n}{3} \,\Big|\, |X_0| > ta_n\Big) + \sum_{i=m}^{r_n} \mathrm{P}\Big(\sum_{k=0}^{i-1}\psi_k |Z_{i-k}| > \frac{ta_n}{3}\Big) \\ & + \mathrm{P}\Big(\phi_1^{m-1}\theta_1 |Z_0| > \frac{ta_n}{3} \,\Big|\, |X_0| > ta_n\Big) \\ \leqslant\; & \frac{\mathrm{P}\Big(|X_0| > \frac{ta_n}{3\phi_1^m}\Big)}{\mathrm{P}(|X_0| > ta_n)} + r_n \mathrm{P}\Big(\sum_{k=0}^{\infty}\psi_k |Z_{-k}| > \frac{ta_n}{3}\Big) + \frac{\mathrm{P}\Big(|Z_0| > \frac{ta_n}{3\phi_1^{m-1}\theta_1}\Big)}{\mathrm{P}(|X_0| > ta_n)} \qquad (4.43) \end{aligned}$$

where (a_n) is a sequence of positive real numbers such that $n\mathrm{P}(|X_0| > a_n) \to 1$ and (r_n) is a sequence of positive integers such that $r_n \to \infty$ and $r_n/n \to 0$ as $n \to \infty$. Since X_0 is regularly varying, the first term on the right hand side in (4.43) converges to $(3\phi_1^m)^\alpha$ as $n \to \infty$. From Theorem 2.3 in Cline [21] we obtain, as $n \to \infty$,

$$\begin{aligned} & r_n \mathrm{P}\Big(\sum_{k=0}^{\infty}\psi_k |Z_{-k}| > \frac{ta_n}{3}\Big) \\ & = \frac{r_n}{n} \cdot n\mathrm{P}\Big(|X_0| > \frac{ta_n}{3}\Big) \cdot \frac{\mathrm{P}\Big(|Z_0| > \frac{ta_n}{3}\Big)}{\mathrm{P}\Big(|X_0| > \frac{ta_n}{3}\Big)} \cdot \frac{\mathrm{P}\Big(\sum_{k=0}^{\infty}\psi_k |Z_{-k}| > \frac{ta_n}{3}\Big)}{\mathrm{P}\Big(|Z_0| > \frac{ta_n}{3}\Big)} \end{aligned}$$

$$\to \; 0 \cdot \Big(\frac{t}{3}\Big)^{-\alpha} \cdot \frac{1}{\sum_{j=0}^{\infty} \psi_j^{\alpha}} \cdot \sum_{j=0}^{\infty} \psi_j^{\alpha} = 0, \tag{4.44}$$

and

$$\frac{\mathrm{P}\Big(|Z_0| > \frac{ta_n}{3\phi_1^{m-1}\theta_1}\Big)}{\mathrm{P}(|X_0| > ta_n)} \to \frac{(3\phi_1^{m-1}\theta_1)^{\alpha}}{\sum_{j=0}^{\infty} \psi_j^{\alpha}}.$$

Hence

$$\limsup_{n\to\infty} \mathrm{P}\Big(\max_{m \leqslant i \leqslant r_n} |X_i| > ta_n \, \Big| \, |X_0| > ta_n \Big) \leqslant 3^{\alpha} (\phi_1^{\alpha})^{m-1} \Big(\phi_1^{\alpha} + \frac{\theta_1^{\alpha}}{\sum_{j=0}^{\infty} \psi_j^{\alpha}} \Big).$$

Letting $m \to \infty$, since $\phi_1^{\alpha} < 1$, it follows

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathrm{P}\Big(\max_{m \leqslant i \leqslant r_n} |X_i| > ta_n \, \Big| \, |X_0| > ta_n \Big) = 0, \qquad t > 0. \tag{4.45}$$

From relation (4.42) we see that

$$X_0 = \phi_1^{|i|} X_i + \sum_{k=0}^{|i|-1} \psi_k Z_{-k} + \phi_1^{|i|-1} \theta_1 Z_i, \qquad i < 0.$$

This leads to

$$\begin{aligned}
&\mathrm{P}\Big(\max_{-r_n \leqslant i \leqslant -m} |X_i| > ta_n \, \Big| \, |X_0| > ta_n \Big) \\
&\quad \leqslant \sum_{i=-r_n}^{-m} \mathrm{P}\Big(|X_i| > ta_n \, \Big| \, |X_0| > ta_n \Big) = \sum_{i=-r_n}^{-m} \frac{\mathrm{P}(|X_i| > ta_n, \, |X_0| > ta_n)}{\mathrm{P}(|X_0| > ta_n)} \\
&\quad \leqslant \sum_{i=-r_n}^{-m} \Bigg[\frac{\mathrm{P}\Big(|X_i| > ta_n, \, \phi_1^{|i|}|X_i| > \frac{ta_n}{3}\Big)}{\mathrm{P}(|X_0| > ta_n)} + \frac{\mathrm{P}\Big(|X_i| > ta_n, \, \sum_{k=0}^{|i|-1} \psi_k |Z_{-k}| > \frac{ta_n}{3}\Big)}{\mathrm{P}(|X_0| > ta_n)} \\
&\qquad\qquad + \frac{\mathrm{P}\Big(|X_i| > ta_n, \, \phi_1^{|i|-1}\theta_1 |Z_i| > \frac{ta_n}{3}\Big)}{\mathrm{P}(|X_0| > ta_n)} \Bigg].
\end{aligned}$$

Since X_i and $(Z_{-k})_{0 \leqslant k \leqslant |i|-1}$ are independent, we obtain the bound

$$\begin{aligned}
&\leqslant \sum_{i=-r_n}^{-m} \frac{\mathrm{P}\Big(|X_0| > ta_n, \, 3\phi_1^{|i|}|X_0| > ta_n\Big)}{\mathrm{P}(|X_0| > ta_n)} + \sum_{i=-r_n}^{-m} \mathrm{P}\Big(\sum_{k=0}^{|i|-1} \psi_k |Z_{-k}| > \frac{ta_n}{3} \Big) \\
&\quad + \sum_{i=-r_n}^{-m} \frac{\mathrm{P}\Big(|X_i| > ta_n, \, 3\phi_i^{|i|-1}\theta_1 |Z_i| > ta_n\Big)}{\mathrm{P}(|X_0| > ta_n)}. \\
&=: \; I_1(n,m) + I_2(n,m) + I_3(n,m).
\end{aligned} \tag{4.46}$$

Take an arbitrary $\epsilon \in (0, \alpha)$. Then using Markov's inequality we obtain

$$\begin{aligned} I_1(n,m) &= \sum_{i=-r_n}^{-m} \frac{\mathrm{P}\Big(3\phi_1^{|i|}|X_0|1_{\{|X_0|>ta_n\}} > ta_n\Big)}{\mathrm{P}(|X_0| > ta_n)} \\ &\leqslant \frac{\mathrm{E}\big[|X_0|^\epsilon 1_{\{|X_0|>ta_n\}}\big]}{(ta_n)^\epsilon \mathrm{P}(|X_0| > ta_n)} \sum_{i=-r_n}^{-m} (3\phi_1^{|i|})^\epsilon. \end{aligned}$$

An application of Theorem 1.12 yields that

$$\limsup_{n\to\infty} I_1(n,m) \leqslant \frac{\alpha}{\alpha-\epsilon} 3^\epsilon \sum_{i=m}^{\infty} (\phi_1^\epsilon)^i,$$

and since $\phi_1^\epsilon < 1$, letting $m \to \infty$, we get

$$\lim_{m\to\infty} \limsup_{n\to\infty} I_1(n,m) = 0.$$

As in (4.44) we obtain

$$\limsup_{n\to\infty} I_2(n,m) \leqslant \limsup_{n\to\infty} r_n \mathrm{P}\Big(\sum_{k=0}^{\infty} \psi_k |Z_{-k}| > \frac{ta_n}{3}\Big) = 0.$$

Since $X_i = \phi_1 X_{i-1} + Z_i + \theta_1 Z_{i-1}$, we have

$$\begin{aligned} I_3(n,m) \leqslant\ & \sum_{i=-r_n}^{-m} \frac{\mathrm{P}\Big(\phi_1|X_{i-1}| > \frac{ta_n}{3},\, 3\phi_i^{|i|-1}\theta_1|Z_i| > ta_n\Big)}{\mathrm{P}(|X_0| > ta_n)} \\ &+ \sum_{i=-r_n}^{-m} \frac{\mathrm{P}\Big(|Z_i| > \frac{ta_n}{3},\, 3\phi_i^{|i|-1}\theta_1|Z_i| > ta_n\Big)}{\mathrm{P}(|X_0| > ta_n)} \\ &+ \sum_{i=-r_n}^{-m} \frac{\mathrm{P}\Big(\theta_1|Z_{i-1}| > \frac{ta_n}{3},\, 3\phi_i^{|i|-1}\theta_1|Z_i| > ta_n\Big)}{\mathrm{P}(|X_0| > ta_n)}. \end{aligned}$$

The second term on the right hand side of this inequality can be treated in a similar way as $I_1(n,m)$. Since Z_i and X_{i-1} are independent and $\phi_1^{m-1} \geqslant \phi_1^{|i|-1}$ for all $i = -r_n, \dots, -m$, it follows that the first term is bounded above by

$$r_n \frac{\mathrm{P}\Big(\phi_1|X_0| > \frac{ta_n}{3}\Big)\mathrm{P}\Big(3\phi_1^{m-1}\theta_1|Z_0| > ta_n\Big)}{\mathrm{P}(|X_0| > ta_n)},$$

which by a standard regular variation argument converges to zero as $n \to \infty$. The same can be done for the third term. Thus

$$\lim_{m\to\infty} \limsup_{n\to\infty} I_3(n,m) = 0,$$

and from (4.46) we therefore have

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathrm{P}\Big(\max_{-r_n \leqslant i \leqslant -m} |X_i| > ta_n \,\Big|\, |X_0| > ta_n \Big) = 0, \qquad t > 0. \tag{4.47}$$

Now relations (4.45) and (4.47) imply condition $\mathcal{AC}(a_n)$.

Let (Y_n) be the tail process of (X_n). Fix $i \in \mathbb{N}$ and let $\epsilon > 0$ be arbitrary. Then from the definition of the tail process in Section 1.3, we have

$$\mathrm{P}(Y_i < -\epsilon,\, Y_0 > 1) = \lim_{x\to\infty} \mathrm{P}(X_i < -\epsilon x,\, X_0 > x \,|\, |X_0| > x).$$

Taking into account relation (4.42), the fact that X_k and $(Z_j)_{j \geqslant k+1}$ are independent and the recursion in (4.38), we get

$$\begin{aligned}
&\mathrm{P}(X_i < -\epsilon a_n,\, X_0 > a_n \,|\, |X_0| > a_n) \\
&\quad = \frac{\mathrm{P}\Big(\phi_1^i X_0 + \sum_{k=0}^{i-1} \psi_k Z_{i-k} + \phi_1^{i-1}\theta_1 Z_0 < -\epsilon a_n,\, X_0 > a_n\Big)}{\mathrm{P}(|X_0| > a_n)} \\
&\quad \leqslant \frac{\mathrm{P}\Big(\sum_{k=0}^{i-1} \psi_k Z_{i-k} + \phi_1^{i-1}\theta_1 Z_0 < -(\epsilon + \phi_1^i) a_n,\, X_0 > a_n\Big)}{\mathrm{P}(|X_0| > a_n)} \\
&\quad \leqslant \frac{\mathrm{P}\Big(\sum_{k=0}^{i-1} \psi_k Z_{i-k} < -\frac{(\epsilon+\phi_1^i)a_n}{2},\, X_0 > a_n\Big)}{\mathrm{P}(|X_0| > a_n)} + \frac{\mathrm{P}\Big(Z_0 < -\frac{(\epsilon+\phi_1^i)a_n}{2\phi_1^{i-1}\theta_1},\, X_0 > a_n\Big)}{\mathrm{P}(|X_0| > a_n)} \\
&\quad \leqslant \mathrm{P}\Big(\sum_{k=0}^{i-1} \psi_k Z_{i-k} < -\frac{(\epsilon+\phi_1^i)a_n}{2}\Big) \\
&\qquad + \frac{\mathrm{P}\Big(Z_0 < -\frac{(\epsilon+\phi_1^i)a_n}{2\phi_1^{i-1}\theta_1},\, \phi_1 X_{-1} + Z_0 + \theta_1 Z_{-1} > a_n\Big)}{\mathrm{P}(|X_0| > a_n)}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \ \mathrm{P}\Big(\Big|\sum_{k=0}^{i-1}\psi_k Z_{i-k}\Big| > \frac{(\epsilon+\phi_1^i)a_n}{2}\Big) + \frac{\mathrm{P}\Big(Z_0 < -\frac{(\epsilon+\phi_1^i)a_n}{2\phi_1^{i-1}\theta_1},\, X_{-1} > \frac{a_n}{3\phi_1}\Big)}{\mathrm{P}(|X_0| > a_n)} \\
&\quad + \frac{\mathrm{P}\Big(Z_0 < -\frac{(\epsilon+\phi_1^i)a_n}{2\phi_1^{i-1}\theta_1},\, Z_0 > \frac{a_n}{3}\Big)}{\mathrm{P}(|X_0| > a_n)} + \frac{\mathrm{P}\Big(Z_0 < -\frac{(\epsilon+\phi_1^i)a_n}{2\phi_1^{i-1}\theta_1},\, Z_{-1} > \frac{a_n}{3\theta_1}\Big)}{\mathrm{P}(|X_0| > a_n)} \\
&\leqslant \ \mathrm{P}\Big(\sum_{k=0}^{\infty}\psi_k |Z_{i-k}| > \frac{(\epsilon+\phi_1^i)a_n}{2}\Big) + \frac{\mathrm{P}\Big(Z_0 < -\frac{(\epsilon+\phi_1^i)a_n}{2\phi_1^{i-1}\theta_1}\Big)\mathrm{P}\Big(X_{-1} > \frac{a_n}{3\phi_1}\Big)}{\mathrm{P}(|X_0| > a_n)} \\
&\quad + \frac{\mathrm{P}\Big(Z_0 < -\frac{(\epsilon+\phi_1^i)a_n}{2\phi_1^{i-1}\theta_1},\, Z_0 > \frac{a_n}{3}\Big)}{\mathrm{P}(|X_0| > a_n)} + \frac{\mathrm{P}\Big(Z_0 < -\frac{(\epsilon+\phi_1^i)a_n}{2\phi_1^{i-1}\theta_1}\Big)\mathrm{P}\Big(Z_{-1} > \frac{a_n}{3\theta_1}\Big)}{\mathrm{P}(|X_0| > a_n)}.
\end{aligned}
$$

A standard regular variation argument (as before in checking condition $\mathcal{AC}(a_n)$) yields that the first, second and forth term on the right hand side of the last inequality converge to zero as $n \to \infty$. Since ϕ_1 and θ_1 are positive, the third term is trivially equal to zero. Hence

$$
\mathrm{P}(X_i < -\epsilon a_n,\, X_0 > a_n \,\big|\, |X_0| > a_n) \to 0, \qquad \text{as } n \to \infty,
$$

and this imply $\mathrm{P}(Y_i < -\epsilon,\, Y_0 > 1) = 0$ for any $\epsilon > 0$. Since $|Y_0| > 1$ a.s., it follows $\mathrm{P}(Y_i < 0,\, Y_0 > 0) = 0$ for any $i > 0$. In the same way we obtain $\mathrm{P}(Y_i > 0,\, Y_0 < 0) = 0$. Hence $(Y_n)_{n \geqslant 0}$ a.s. has no two values of the opposite sign. The same conclusion can be obtained for $(Y_n)_{n \leqslant 0}$, thus yielding the tail process $(Y_n)_{n \in \mathbb{Z}}$ a.s. has no two values of the opposite sign.

Assume that the sequence (X_n) is strongly mixing. One sufficient condition for this property in our case is that the characteristic function φ_1 of Z_1 is integrable and satisfies

$$
\int |\varphi_1(x)|\, dx \leqslant 2\pi,
$$

see Chanda [19] (some other sufficient conditions for the strong mixing property of (X_n) can be found in Athreya and Pantula [2] and Mokkadem [52]). Then the mixing condition $\mathcal{A}'(a_n)$ holds. If Condition 2.14 holds when $\alpha \in [1,2)$, then all conditions in Theorem 2.15 are satisfied and we obtain the functional limit theorem for the ARMA(1,1) model considered in this section. With the same explanation as in Section 4.1 and with an additional assumption that $\sum_{j=0}^{\infty} \psi_j^{\alpha} = 1$, we have that for $\alpha \in (0,1) \cup (1,2)$, the characteristic triple of the limiting process is of the form $(0, \nu, b)$, where

$$\nu(dx) = \alpha \bigg(\sum_{j=0}^{\infty} \psi_j \bigg)^{\alpha} |x|^{-1-\alpha} \Big(p 1_{(0,\infty)}(x) + q 1_{(-\infty,0)}(x) \Big) dx, \tag{4.48}$$

$$b = (p-q) \frac{\alpha}{1-\alpha} \bigg[\Big(\sum_{j=0}^{\infty} \psi_j \Big)^{\alpha} - 1 \bigg]. \tag{4.49}$$

Hence we have proven the following result.

Proposition 4.3. *Let (X_n) be an ARMA(1,1) process, i.e.*

$$X_n = \phi_1 X_{n-1} + Z_n + \theta_1 Z_{n-1},$$

where the coefficients ϕ_1, θ_1 are positive and $\phi_1 < 1$, and the sequence (Z_n) consists of i.i.d. regularly varying random variables with index $\alpha \in (0,2)$ such that $\mathrm{E} Z_1 = 0$ if $\alpha \in (1,2)$. Suppose (X_n) is strongly mixing and that Condition 2.14 holds when $\alpha \in [1,2)$. Then the following statements hold.

1. *The partial sum process $V_n(\,\cdot\,)$, defined by (2.4), converges in distribution to an α–stable Lévy process $V(\,\cdot\,)$ in $D[0,1]$ under the M_1 topology.*

2. *Assume $\alpha \in (0,1) \cup (1,2)$ and $\sum_{j=0}^{\infty} \psi_j^{\alpha} = 1$. Then the characteristic triple of the limiting process $V(\,\cdot\,)$ is of the form $(0, \nu, b)$, where ν and b are given in (4.48) and (4.49).*

4.4 Stochastic volatility models

Consider the stochastic volatility model

$$X_n = \sigma_n Z_n, \qquad n \in \mathbb{Z},$$

where the noise sequence (Z_n) consists of i.i.d. regularly varying random variables with index of regular variation $\alpha \in (0,2)$, and the volatility sequence (σ_n) is strictly stationary, independent of the sequence (Z_n) and consists of positive random variables such that $\mathrm{E}(\sigma_0^{2\alpha+r}) < \infty$ for some $r > 0$.

An application of the well known Breiman's lemma [17, Proposition 3] on regularly varying tail of a product of two independent random variables yields that every random variable X_n is regularly varying with index α. Assume further that $(\ln \sigma_n)_n$ is a Gaussian causal ARMA process. Then (X_n) is strongly mixing with geometric rate (see Davis and Mikosch [26]).

Take $0 < s < \min\{r, 4-2\alpha\}$. For any $i \in \mathbb{N}$ and $x > 0$ we have by Markov's inequality and the fact that the sequence (Z_n) is i.i.d.,

$$\begin{aligned}
\mathrm{P}(|X_i| > xa_n, |X_0| > xa_n) &\leqslant \mathrm{P}(\max\{\sigma_i, \sigma_0\} \cdot \min\{|Z_i|, |Z_0|\} > xa_n) \\
&\leqslant \mathrm{P}(\max\{\sigma_i, \sigma_0\} > (xa_n)^{1/2-s/8}) + \mathrm{P}(\min\{|Z_i|, |Z_0|\} > (xa_n)^{1/2+s/8}) \\
&\leqslant (xa_n)^{-(1/2-s/8)(2\alpha+s)} \mathrm{E}[(\max\{\sigma_i, \sigma_0\})^{2\alpha+s}] + [\mathrm{P}(|Z_0| > (xa_n)^{1/2+s/8})]^2 \\
&\leqslant (xa_n)^{-(1/2-s/8)(2\alpha+s)} \cdot 2\mathrm{E}(\sigma_0^{2\alpha+s}) + [\mathrm{P}(|Z_0| > (xa_n)^{1/2+s/8})]^2,
\end{aligned}$$

where (a_n) is a sequence of positive real numbers such that $n\mathrm{P}(|X_1| > a_n) \to 1$ as $n \to \infty$. Take $0 < p < \min\{s/8, s(4-2\alpha-s)/8\alpha\}$ and define $r_n = \lfloor n^p \rfloor$. Then

$$\begin{aligned} n \sum_{i=1}^{r_n} \mathrm{P}(|X_i| > xa_n, |X_0| > xa_n) &\leqslant 2\mathrm{E}(\sigma_0^{2\alpha+s}) n r_n (xa_n)^{-(1/2-s/8)(2\alpha+s)} \\ &\quad + n r_n [\mathrm{P}(|Z_0| > (xa_n)^{1/2+s/8})]^2 \\ &=: I_1(n) + I_2(n). \end{aligned} \tag{4.50}$$

Since $a_n = n^{1/\alpha} L'(n)$ for some slowly varying function L', we have

$$I_1(n) \leqslant C_1 n^{1+p-(1/2-s/8)(2+s/\alpha)} L_1'(n),$$

where $C_1 = C_1(x) = 2\mathrm{E}(\sigma_0^{2\alpha+s}) x^{-(1/2-s/8)(2\alpha+s)}$ and $L_1'(n) = (L'(n))^{-(1/2-s/8)(2\alpha+s)}$. From the definition of p we obtain $1+p-(1/2-s/8)(2+s/\alpha) < 0$, which by Proposition 1.3.6 in Bingham et al. [13] implies $I_1(n) \to 0$ as $n \to \infty$.

Take now $0 < k < \alpha s/(4+s)$. Since $\mathrm{P}(|Z_0| > x) = x^{-\alpha} L(x)$ for some slowly varying function L, we have

$$\begin{aligned} I_2(n) &\leqslant C_2 n^{1+p} (a_n)^{(1/2+s/8)(-2\alpha+k)} (a_n)^{-(1/2+s/8)k} \, [L((xa_n)^{(1/2+s/8)})]^2 \\ &= C_2 n^{1+p+(1/2+s/8)(-2+k/\alpha)} L_2'(n) \cdot c_n, \end{aligned}$$

where $C_2 = C_2(x) = x^{-2\alpha(1/2+s/8)}$, $L_2'(n) = (L'(n))^{(1/2+s/8)(-2\alpha+k)}$, and

$$c_n = (a_n)^{-(1/2+s/8)k} \, [L((xa_n)^{(1/2+s/8)})]^2.$$

From the definitions of p and k it follows that $1+p+(1/2+s/8)(-2+k/\alpha) < 0$. Since also $-(1/2+s/8)k < 0$, from Proposition 1.3.6 in [13] we obtain

$$n^{1+p+(1/2+s/8)(-2+k/\alpha)} L_2'(n) \to 0 \quad \text{and} \quad c_n \to 0,$$

and hence $I_2(n) \to 0$ as $n \to \infty$. Therefore from relation (4.50) it follows

$$\lim_{n\to\infty} n \sum_{i=1}^{r_n} \mathrm{P}(|X_i| > xa_n, |X_0| > xa_n) = 0, \qquad x > 0.$$

Hence relation (3.18) holds.

Therefore, we can apply Theorem 3.7 to obtain the following proposition.

Proposition 4.4. *Let (X_n) be a stochastic volatility model, i.e. $X_n = \sigma_n Z_n$, where (Z_n) is a i.i.d. sequence of regularly varying random variables with index $\alpha \in (0, 2)$, and the sequence (σ_n) is independent of the sequence (Z_n) and consists of positive random variables such that $\mathrm{E}(\sigma_1^{2\alpha+r}) < \infty$ for some $r > 0$. Assume $(\ln \sigma_n)$ is a Gaussian causal ARMA process and that Condition 2.14 holds for $\alpha \in [1, 2)$. Then the partial sum stochastic process $V_n(\,\cdot\,)$, defined by (2.4), converges in distribution, to an α-stable Lévy process $V_0(\,\cdot\,)$ in $D[0, 1]$ under the J_1 topology. The characteristic triple of the limiting process $V_0(\,\cdot\,)$ is of the form $(0, \mu, 0)$, where the measure μ is the vague limit of $n\mathrm{P}(X_1/a_n \in \cdot)$ as $n \to \infty$.*

We recall here that one sufficient condition for Condition 2.14 to hold is that the sequence (X_n) is a function of a Gaussian causal ARMA process, i.e. $X_n = f(A_n)$, for some Borel function $f : \mathbb{R} \to \mathbb{R}$ and some Gaussian causal ARMA process (A_n). From the results in Brockwell and Davis [18] and Pham and Tran [57] (see also Davis and Mikosch [26]) it follows that the sequence (A_n) is strongly mixing with geometric rate. In this particular case this implies that the sequence (A_n) is ρ-mixing with geometric rate (see Kolmogorov and Rozanov [43, Theorem 2]), a property which transfers immediately to the series (X_n). Hence by Proposition 2.19, Condition 2.14 holds.

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Summary

Functional limit theorems present a rich and interesting part of probability theory. They have been first obtained for independent and identically distributed random variables with finite second moments. This is the content of Donsker's theorem (see for instance Billingsley [12]). One direction of extending these results is to replace the independence by weak dependence, for example by assuming the underlying random variables are strongly mixing. The other direction involves studying functional limit theorems for random variables with infinite second moments. It is well known that regularly varying random variables with tail index $\alpha \in (0, 2)$ have infinite second moments. This thesis is dedicated to both of these two extensions.

More precisely, let $(X_n)_{n \geqslant 1}$ be a strictly stationary sequence of random variables. This thesis investigates the asymptotic distributional behavior of the partial sum stochastic processes

$$V_n(t) = a_n^{-1}(S_{\lfloor nt \rfloor} - \lfloor nt \rfloor b_n), \qquad t \in [0, 1],$$

under the properties of weak dependence and regular variation with index $\alpha \in (0, 2)$, where $S_n = X_1 + \cdots + X_n$, (a_n) is a sequence of positive real numbers such that, as $n \to \infty$,

$$n \, \mathrm{P}(|X_1| > a_n) \to 1,$$

and

$$b_n = \mathrm{E}\big(X_1 \, 1_{\{|X_1| \leqslant a_n\}}\big).$$

The stochastic processes that we study have discontinuities, so for the underlying function space of their sample paths we choose the space $D[0, 1]$ of all right-continuous real valued functions on $[0,1]$ with left limits. If the $D[0,1]$-valued process $V_n(\,\cdot\,)$ converges in distribution, we say that the sequence (X_n) satisfies the functional limit theorem with respect to a certain metric (or topology) on $D[0,1]$. The most frequently used topology on $D[0,1]$ is Skorohod's J_1 topology. We present three cases for which functional limit theorems hold with respect to this topology. But there are examples when the J_1 topology is not suitable for describing the convergence in distribution of the partial sum stochastic processes. If we use Skorohod's M_1 topology (which is weaker than J_1 topology), then in part of these "problematic" examples we are able to recover the convergence in distribution of the processes $V_n(\,\cdot\,)$ and functional limit theorems will hold.

A special attention is given to the theory of point processes, since they are the base for our results. The convergence in distribution of the processes $V_n(\,\cdot\,)$ is obtained from a new convergence of a special type of time-space point processes through the use of continuous mapping theorem. A notion that is used throughout the whole thesis is regular variation. Beside point processes and regular variation, other major notions that we use are vague convergence, tail process and strong mixing.

The main result of the thesis gives conditions under which a strictly stationary, regularly varying sequence of random variables with index $\alpha \in (0,2)$ satisfies the functional limit theorem with respect to Skorohod's M_1 topology, with the limit being an α–stable Lévy process, which is characterized in terms of its characteristic triple.

We also investigate conditions under which four applied time series models, namely MA, squared GARCH(1,1), ARMA(1,1) and stochastic volatility models, satisfy the functional limit theorem.

Sažetak

Fuunkcionalni granični teoremi predstavljaju bogato i zanimljivo područje teorije vjerojatnosti. Prvo su bili dobiveni za slučaj nezavisnih i jednako distribuiranih slučajnih varijabli koje imaju konačne druge momente. To je sadržaj Donskerovog teorema (vidi Billingsley [12]). Jedan smjer u poopćenju ovih rezultata jest zamjena svojstva nezavisnosti slabom zavisnošću, na primjer pomoću pretpostavke jakog miješanja. Drugi mogući smjer poopćenja uključuje proučavanje funkcionalnih graničnih teorema za slučajne varijable s beskonačnim drugim momentima. Poznato je da regularno varirajuće slučajne varijable s indeksom $\alpha \in (0, 2)$ imaju beskonačne druge momente. Ova disertacija se bavi sa oba smjera.

Preciznije, neka je $(X_n)_{n \geqslant 1}$ strogo stacionaran niz slučajnih varijabli. Ova disertacija istražuje asimptotsko ponašanje distribucija slučajnih procesa parcijalnih suma

$$V_n(t) = a_n^{-1}(S_{\lfloor nt \rfloor} - \lfloor nt \rfloor b_n), \qquad t \in [0, 1],$$

uz uvjete slabe zavisnosti i regularne varijacije sa indeksom $\alpha \in (0, 2)$, gdje je $S_n = X_1 + \cdots + X_n$, (a_n) niz pozitivnih realnih brojeva takav da, kada $n \to \infty$,

$$n \, \mathrm{P}(|X_1| > a_n) \to 1,$$

i

$$b_n = \mathrm{E}\big(X_1 \, 1_{\{|X_1| \leqslant a_n\}}\big).$$

Slučajni procesi koje proučavamo imaju prekide, pa za funkcijski prostor njihovih putova koristimo prostor $D[0, 1]$ svih zdesna neprekidnih realnih funkcija na $[0, 1]$ sa limesima slijeva. Ako process $V_n(\,\cdot\,)$ konvergira po distribuciji u $D[0, 1]$, kažemo da

niz (X_n) zadovoljava funkcionalan granični teorem obzirom na pripadnu metriku (ili topologiju) na $D[0,1]$. Najčešće korištena topologija na $D[0,1]$ je Skorohodova J_1 topologija. Tri primjera u kojima funkcionalni granični teoremi vrijede obzirom na ovu topologiju prezentirana su u disertaciji. Postoje primjeri kada J_1 topologija nije pogodna za opis konvergencije po distribuciji slučajnih procesa parcijalnih suma. No ako upotrijebimo Skorohodovu M_1 topologiju (koja je slabija od J_1 topologije), tada smo u dijelu tih "problematičnih" primjera u mogućnosti dobiti konvergenciju po distribuciji slučajnih procesa $V_n(\,\cdot\,)$ i funkcionalni granični teoremi će vrijediti.

Poseban je naglasak stavljen na teoriju točkovnih procesa, koji predstavljaju temelj dobivenih rezultata. Konvergencija po distribuciji slučajnih procesa $V_n(\,\cdot\,)$ je dobivena iz nove konvergencije jednog specijalnog niza točkovnih procesa korištenjem teorema o neprekidnom preslikavanju. Pojam koji se koristi kroz cijelu disertaciju je regularna varijacija. Osim točkovnih procesa i regularne varijacije, ostali važni pojmovi koje koristimo jesu slabašna [vague] konvergencija, repni proces i jako miješanje.

Glavni rezultat disertacije navodi uvjete pod kojima strogo stacionaran, regularno varirajući niz slučajnih varijabli s indeksom $\alpha \in (0,2)$ zadovoljava funkcionalni granični teorem obzirom na Skorohodovu M_1 topologiju, s α–stabilnim Lévyjevim procesom kao limesom, koji je karakteriziran pomoću svoje karakteristične trojke.

Na kraju disertacije istražujemo uvjete pod kojima četiri modela vremenskih nizova, koji se često koriste u primjenama, zadovoljavaju funkcionalni granični teorem.

Curriculum Vitae

Danijel Krizmanić was born in Pula, Croatia, on December 24, 1979. After having received elementary (*Osnova škola V. Nazor Pazin, PRO Karojba*) and intermediate education (*Gimnazija J. Dobrila Pazin*), he attended the University of Zagreb, where he obtained a undergraduate degree in Mathematics in 2003. His master thesis was titled: *Basic Probabilistic Results in Sequential Analysis.* In 2004 he was employed as a teaching assistant at the Department of Mathematics, University of Rijeka. In the same year he became a Ph.D. student at the Department of Mathematics, University of Zagreb. His research interests are mostly in probability theory, particularly regular variation and functional limit theorems.

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