Finite geometry, designs, codes, and Hamada's conjecture

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Outline

Designs

A t-(v, k, λ) design $\mathcal{D}$=$(\mathcal{X}, \mathcal{B})$ is a set $\mathcal{X}$ of points and a collection $\mathcal{B}$ of subsets of $\mathcal{X}$ called blocks such that:

- $|\mathcal{X}| = v$,
- $|B| = k$ for each $B \in \mathcal{B}$, and
- Every t-subset of $\mathcal{X}$ s contained in exactly λ blocks.

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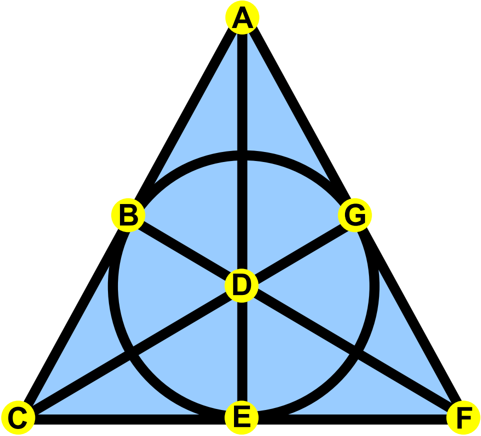
A small example



A 2-$(7, 3, 1)$ design

Properties

The t-designs are highly regular:

- If $0 \leq i \leq t$, any i-subset appears in $\lambda_i = \lambda\binom{v-i}{t-i}/\binom{k-i}{t-i}$ blocks.
- $\mathbf{i = 0}$: Total number of blocks is $b = \lambda\binom{v}{t}/\binom{k}{t}$
- $\mathbf{i = 1}$: Any point x appears in r blocks, where

$$r = \lambda_1 = \lambda\binom{v-1}{t-1}/\binom{k-1}{t-1}.$$

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Incidence Matrices

The incidence matrix of a t-(v, k, λ) design is a $b \times v$ $(0, 1)$ matrix whose (i, j) entry is 1 if block i contains point j, and 0 otherwise.

The 2-(7,3,1) Design:

	A	B	C	D	E	F	G
B_1	1	1	1	0	0	0	0
B_2	1	0	0	1	1	0	0
B_3	1	0	0	0	0	1	1
B_4	0	1	0	1	0	1	0
B_5	0	1	0	0	1	0	1
B_6	0	0	1	1	0	0	1
B_7	0	0	1	0	1	1	0

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Finite Geometries

Projective Geometry $PG(n, q)$

- **points** are the 1-dimensional subspaces of $\mathbb{F}_q^{n+1}$.
- **lines** are the 2-dimensional subspaces of $\mathbb{F}_q^{n+1}$
- k**-dimensional subspaces** are the $(k+1)$-dimensional subspaces of $\mathbb{F}_q^{n+1}$.

Affine Geometry $AG(n, q)$

- **points** are the vectors of $\mathbb{F}_q^n$
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Geometric Designs

A geometric design is formed from the points and d-subspaces of $PG(n, q)$ or $AG(n, q)$.

The projective geometry design $PG_d(n, q)$:

$$2 - \left(\frac{q^{n+1} - 1}{q - 1}, \frac{q^{d+1} - 1}{q - 1}, \frac{(q^{n+1} - q^2)(q^{n+1} - q^3) \cdots (q^{n+1} - q^d)}{(q^{d+1} - q^2)(q^{d+1} - q^3) \cdots (q^{d+1} - q^d)} \right)$$

The affine geometry design $AG_d(n, q)$:

$$2 - \left(q^n, q^d, \frac{(q^n - q)(q^n - q^2) \cdots (q^n - q^{d-1})}{(q^d - q)(q^d - q^2) \cdots (q^d - q^{d-1})} \right)$$

If $q = 2$, $AG_d(n, 2)$ is also a 3-$(2^n, 2^d, \frac{(2^n - 2^2) \cdots (2^n - 2^{d-1})}{(2^d - 2^2) \cdots (2^d - 2^{d-1})}$ design.

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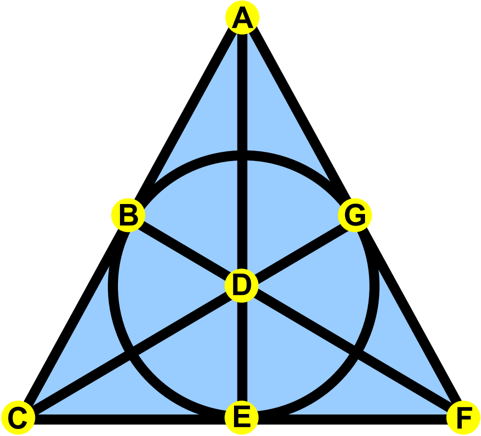
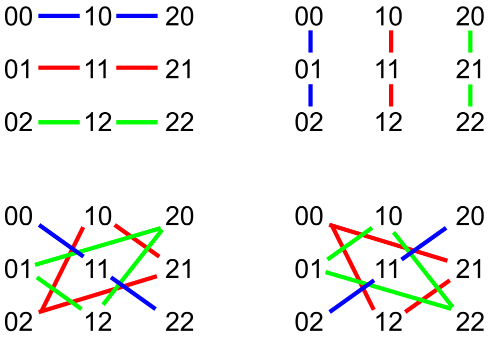
A small examplei: $PG_1(2,2)$



$PG_1(2,2)$: The projective plane of order 2

Affine Geometry Designs are Resolvable

$AG_1(2,3)$, or a 2-$(9,3,1)$-design



This design is resolvable into **parallel classes.**

Linear error-correcting codes

Linear code

A **linear** q**-ary** $[n, k, d]$ **code** C is a k-dimensional subspace of the n-dimensional vector space over the field $GF(q)$ of order q with minimum Hamming distance d.

A code with minimum distance d can correct up to $e = [(d-1)/2]$ errors.

Dual code

The dual code $C^{\perp}$ of an $[n, k]$ code C is the $[n, n-k]$ code defined by

$$C^{\perp} = \{y \in GF(q)^n \mid y \cdot x = 0 \text{ for all } x \in C\}$$

Parity check matrix

A matrix H of q-rank $n-k$ whose rows are vectors from $C^{\perp}$ is a parity check matrix of C.

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Majority logic decoding algorithm

If a codeword $x = (x_1, \ldots, x_n) \in C$ is sent over a communication channel, and a vector $y = (y_1, \ldots, y_n)$ is received, for each coordinate i, $1 \leq i \leq n$, the values

$$y_i^{(1)}, \ldots, y_i^{(r_i)} \tag{1}$$

of r_i linear functions are computed, and y_i is decoded as the most frequent among the values (1).

Theorem. (Rudolph, 1967)

If C is a linear $[n, k]$ code such that $C^{\perp}$ contains a set **S** of vectors of weight w whose supports are the blocks of a 2-(n, w, λ) design, the code C can correct up to

$$e = \left\lfloor \frac{r + \lambda - 1}{2\lambda} \right\rfloor$$

errors by majority logic decoding, where $r = \lambda_1 = \lambda(n-1)/(w-1)$.

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Sketch of proof.

If $a = (a_1, \ldots, a_n) \in \mathbf{S}$ then

$$a_1 x_1 + \cdots + a_n x_n = 0$$

for every $x \in C$.

Note

Due to possible errors in the received vector $y = (y_1, \ldots, y_n)$,

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and $a_i \neq 0$. Then

$$y_i = -\frac{a_1}{a_i} y_1 - \cdots - \frac{a_n}{a_i} y_n.$$

Linear functions f_j for decoding y_i:

For each i, $1 \leq i \leq n$, the set **S** contains r vectors

$$a^{(j)} = (a_1^{(j)}, \ldots, a_n^{(j)}),\ j = 1, \ldots, r$$

such that $a_i^{(j)} \neq 0$.
We define a set of $r + \lambda$ linear functions $f_j = f_j(y_1, \ldots, y_n)$,

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Linear functions f_j for decoding y_i:

For each i, $1 \leq i \leq n$, the set **S** contains r vectors

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Thus, if there are **e** errors in $y = (y_1, \ldots, y_n)$, and

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the majority of the values $f_j(y_1, \ldots, y_n)$ will be equal to x_i. □

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- **Rahman and Blake**, 1975:
  Rudolph's bound can be improved if If $C^{\perp}$ supports a t-design with $t \geq 2$.
- If $t = 1$, λ can be replaced with the maximum frequency of appearance of pairs of points.
- If $t = 0$, r can be replaced with the minimum frequency of appearance of a point in blocks.

Milti-step majority logic decoding

Rudolph's algorithm is an example of one-step majority logic decoding.

There is an iterative multistep version of the algorithm consisting of a sequence of one-step decoding of linear combinations of received bits, followed by computing the individual bits y_i as a solution of a system of linear equations.

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Which codes support t-designs?

Task

Find a linear code C so that $C^{\perp}$ supports a t-design with $t \geq 2$.

The Assmus-Mattson Theorem, 1969

If C is a linear $[n, k]$ code with minimum distance d such that the number of distinct nonzero weights in $C^{\perp}$ not exceeding $n - t$ is smaller than $d - t$, then both C and $C^{\perp}$ support t-designs.

Note

The Assmus-Mattson Theorem gives a sufficient condition for the existence of designs in a code.
It does not specify how one can find such codes.

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Codes with mulitransitive automorphism groups

If C admits an automorphism group of permutations that acts t-transitive (or t-homogeneously) on the set of n code coordinates, then the supports of all codewords of any nonzero weight form a t-design.

Example

The binary Golay $[24, 12, 8]$ code and the ternary Golay $[12, 6, 6]$ code support 5-designs.

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A simple construction using incidence matrices

If C is a linear code over $GF(q)$ of length v with a parity check matrix H being the block by point $b \times v$ incidence matrix of a t-(v, w, λ) design D, then $C^{\perp}$ supports the t-(v, w, λ) design D.
The dimension of C is $k = v - rank_q H$.

A possible drawback:

Fisher inequality

If D is a t-(v, w, λ) design with b blocks such that $t \geq 2$ and $v > w > 0$, then

$$b \geq v.$$

Thus, it can happen that $rank_q H = v$ and $dim(C) = 0$.

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Designs with minimum p-rank

Task

Given $v > w > 0$, $\lambda > 0$, and a prime p such that $p | r - \lambda$, find a 2-(v, w, λ) design of minimum p-rank.

Example

Let $v = 8, \ w = 4, \ \lambda = 3.$

Then $r = 7$, $r - \lambda = 7 - 3 = 4$, and $p = 2 | (r - \lambda)$.

There exist four non-isomorphic 2-$(8, 4, 3)$ designs, and their 2-ranks are 4, 5, 6, and 7 respectively.

Note

The 2-$(8, 4, 3)$ design of minimum 2-rank, 4, is isomorphic to the geometric design $AG_2(3, 2)$.

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Two fundamental questions

Given parameters $v > w > 0$, $\lambda > 0$, such that a 2-(v, w, λ) design exists,

- What is the minimum p-rank of a 2-(v, w, λ) design?
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p-Ranks of Geometric Designs

The p-ranks of the geometric designs were computed in the 1960's and 1970's.

Theorem. (Graham and MacWilliams '66, Weldon '67)

For any prime $p \geq 2$, and any integer $s \geq 1$,

$$rank_p PG_1(2, p^s) = \binom{p+1}{2}^s + 1.$$

Theorem. (Sachar '79)

If Π is a projective plane of prime order p (a 2-$(p^2+p+1, p+1, 1)$ design) then

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Theorem. (Hamada '73)

(a) The p-rank of $PG_d(n, p^s)$ is given by

$$\sum_{t_0,\ldots,t_s} \prod_{j=0}^{s-1} \sum_{i=0}^{[(t_{j+1}p-t_j)/p]} (-1)^i \binom{n+1}{i} \binom{n+t_{j+1}p-t_j-ip}{n},$$

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Corollary

$$rank_2 AG_d(n,2) = \sum_{i=0}^{n-d} \binom{n}{i}.$$

Note

The binary code spanned by the incidence matrix of $AG_d(n,2)$ is equivalent to the **Reed-Muller** code of length 2^n and order d.

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A q-ary linear code spanned by the incidence matrix of $PG_d(n,q)$ or $AG_d(n,q)$ is a **finite geometry code**.

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Conjecture (Hamada, 1973) : A geometric design over $\mathbb{F}_{p^m}$ has minimum p-rank among all designs with the given parameters.

Example

Let $v = 8, \; w = 4, \; \lambda = 3.$

There exist exactly four non-isomorphic 2-$(8, 4, 3)$ designs, and their 2-ranks are 4, 5, 6, and 7 respectively.

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The only 2-$(8, 4, 3)$ design of minimum 2-rank is isomorphic to the geometric design $AG_2(3, 2)$.

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The Proven Cases

Hamada's Conjecture has been proved in the following cases:

- **Hamada and Ohmori (1975):** True for $PG_{n-1}(n, 2)$ and $AG_{n-1}(n, 2)$.
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with equality if and only if D is isomorphic to the **complementary design** of $PG_{n-1}(n,2)$.
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A revised version of Hamada's Conjecture

Generalized Incidence Matrix (T. '99)

A generalized incidence matrix of a design has entries in $GF(q)$, with nonzero entries designating incidence.

Definition

The dimension of a design D over $GF(q)$, ($dim_q(D)$), is defined as the minimum q-rank of all generalized incidence matrices of D over $GF(q)$.

Example

The 3-rank of the $(0,1)$-incidence matrix of the unique 5-$(12,6,1)$ design D_{12} is 11, while $dim_3(D_{12}) = 6$.

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Let q be an arbitrary prime power, and let $n \geq 2$.
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Let D be a 2-$(121, 100, 99)$ design. Then

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with equality $dim_{11}(D) = 3$ if and only if D is isomorphic to the **complementary design** of the Desarguesian affine plane of order 11.

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Are geometric designs characterized by their p-rank?

Question

Are geometric designs characterized as the unique designs with the given parameters and p-rank?

Answer

Yes, in all proved cases of Hamada's Conjecture.

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Non-geometric designs with the same p-rank as geometric ones

There are known **non-geometric designs** having the same parameters and the same p-rank as certain **geometric** designs:

- Untill recently, all known such designs were

$$2-(31,7,7), 3-(32,8,7), (p=2);\ 2-(64,16,5),\ (q=2^2).$$

- In 2008 and 2009, infinitely many designs were found with for arbitrary **prime** $p \geq 2$.

These designs indicate that although geometric designs may have minimum p-rank, they are not always the **unique** designs of minimum p-rank.

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Non-geometric designs of minimum p-rank

Deigns from self-dual codes

Theorem (T '86).
(i) In addition to $PG_2(4,2)$, there are four non-geometric 2-$(31,7,7)$ designs with block intersection numbers $\{1,3\}$, all having 2-rank 16.
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Proof

Use Rudolph's theorem, the Assmus-Mattson theorem, and the classification of binary self-dual $[32,16,8]$ codes.

Quasi-symmetric design

A design with two distinct block intesection numbers.

Note

Two 2-$(31,7,7)$ designs supported by the projective geometry code and the QR code were mentioned by Goethals and Delsarte in 1968.

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Designs from Nets

Symmetric (μ, m)-Nets

A **symmetric** (μ, m)**-net** is a 1-$(m^2\mu, m\mu, m\mu)$ design D such that both D and its dual design D^* are uniquely resolvable ito parallel classes of size m, so that any non-parallel blocks share exactly μ points .

Class-regular nets

A symmetric (μ, m)-net is **class-regular** if it admits an automorphism group of order m that acts transitively on each block and point parallel class.

The classical (q, q)-net

Points and planes of $AG(3, q)$ that do not contain lines from a given parallel class.

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The class-regular $(4,4)$ nets and their codes

The $(4,4)$ nets

A $(4,4)$-net consists of 64 points and 64 blocks, each block of size 16 and each point in 16 blocks, so that the blocks (as well as and points) are partitioned into 16 parallel classes of size 4, and any two non-parallel blocks share 4 points.

Theorem. (Harada, Lam & T., 2005)

(i) Up to isomorphism, there are exactly 239 class-regular $(4,4)$-nets.
(ii) The minimum 2-rank of a $(4,4)$-net is 16.
(iii) The binary codes of **three** $(4,4)$-nets support 2-$(64,16,5)$ designs of 2-rank 16:

- The code of the classical $(4,4)$-net supports $AG_2(3,2^2)$.
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Non-geometric designs from line spreads

Theorem. (Mavron, McDonough, & T., 2008)

One of the non-geometric 2-$(64, 16, 5)$ designs of 2-rank 16 found by Harada, Lam and T., can be btained from a (very special) line spread of $PG(5, 2)$.

Theorem. (Mateva and Topalova, 2008)

- There are 131,044 inequivalent line spreads of $PG(5, 2)$.
- Two of these line spreads yield 2-$(64, 16, 5)$ designs of 2-rank 16: $AG_2(3, 2^2)$, and the non-geometric design found by Harada, Lam, T., and Mavron, McDonough and T.

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- Two of these line spreads yield 2-$(64, 16, 5)$ designs of 2-rank 16: $AG_2(3, 2^2)$, and the non-geometric design found by Harada, Lam, T., and Mavron, McDonough and T.

Non-geometric designs from line spreads

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Designs from Polarities in $PG(n, q)$

The motivating example

The geometric design $PG_2(4, 2)$ and one of the non-geometric 2-$(31, 7, 7)$ designs of 2-rank 16 share the following structure:

$2-(15, 7, 3)$ Planes $\in PG(3, 2)$	$2-(15, 3, 1) \times 4$ Lines $\in PG(3, 2)$
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Polarities in $PG(n,q)$

A **polarity** α of $PG(n,q)$ is an involutory isomorphism between $PG(n,q)$ and its dual space:

$$\begin{array}{rcl} \alpha: \text{ point} & \longleftrightarrow & \text{hyperplane}, \\ & \cdots & \\ \text{i}-\text{subspace} & \longleftrightarrow & \text{(n-1-i)}-\text{subspace} \\ & \cdots & \end{array}$$

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Let α be a polarity of $PG(3,q)$:

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A new class of quasi-symmetric designs from polaities in $PG(4, q)$

Theorem. (Jungnickel & T., 2008)

Permuting the lines of a hyperplane $H = PG(3, q) \subset PG(4, q)$ via a polarity α of H transforms $PG_2(4, q)$ into another **non-geometric** quasi-symmetric design with intersection numbers $\{1,\ q+1\}$.

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Lines of $PG(4, q)$ which meet $H = PG(3, q)$ in one point are transformed by α into "lines" of size 2.

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$$PG_k(2k,q)\left\{\begin{array}{|c|c|}\hline PG_k(2k-1,q) & PG_{k-1}(2k-1,q) \\ \hline \emptyset & AG_k(2k,q) \\ \hline \end{array}\right.$$

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Any polarity α of $PG(2k-1, q)$ maps any $(k-1)$-subspace to a $(k-1)$-subspace.

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The p-rank of a design obtained via polarity

Theorem. (Jungnickel & T., 2008)

Let α be a polarity of $PG(2k-1,q)$, where $q = p^s$ and p is a prime, and let $\alpha(D)$ be the design obtained from $PG_k(2k,q)$. Then

$$rank_p PG_k(2k,q) \leq rank_p \alpha(D) \leq \frac{1}{2}\Big(\frac{q^{2k+1}-1}{q-1}+1\Big).$$

If $q = p$ is a **prime** then

$$rank_p PG_k(2k,q) = rank_p \alpha(D).$$

An example of a non-prime q

If $q = 4 = 2^2$ and $k = 2$, we have

$$rank_2 PG_2(4,4) = 146 < 154 = rank_2 \alpha(D) < \frac{4^5-1}{4-1} = 171.$$

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Hamada's formula for $r_p = rank_p PG_k(2k, p)$, p prime,
as simplified by **Hirschfeld and Shaw '94**, is

$$r_p = \frac{p^{2k+1}-1}{p-1} - \sum_{i=0}^{k-1}(-1)^i \binom{(k-i)(p-1)-1}{i}\binom{k+(k-i)p}{2k-i}. \quad (2)$$

What we need is

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The expressions (2) and (3) are equal:

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The following identity holds for any positive **integer** p:

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A Generalization to the Affine Case

Let H be a hyperplane of $AG(n, q)$.
A d-dimensional subspace L of $AG(n, q)$, $d \leq n - 1$, is either

- disjoint from H, or
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Cross Block

We call L a **cross block** if $\dim(L \cap H) = d - 1$.

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An Affine Space "Distortion" Construction:

Let $D = AG_d(n, q)$.

- Fix a hyperplane H through 0 in $AG(n, q)$.
- Fix a permutation α of the $(d-1)$-spaces through 0 in H.
- Replace each cross block $B = B_{out} \cup B_{in}$ containing 0 with $\alpha(B) = B_{out} \cup \alpha(B_{in})$.
- Replace each coset of B with a **carefully chosen** coset of $\alpha(B)$.
- If $q = 2$, we must similarly "distort" all other blocks B' such that $B'_{in} = B_{in}$.

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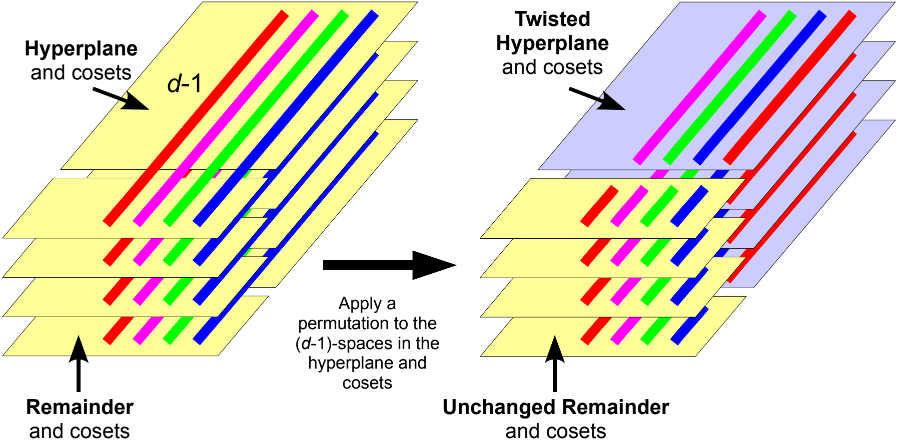
Affine construction



Affine Construction: Details

What is a "carefully chosen" coset of $\alpha(B)$?

- $\alpha(B)$ is not a vector subspace any longer.
- There are q^{n-d} cosets of B by elements of H.
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- Choose q^{n-d} elements of H so that each coset of B and each coset of $\alpha(B_{in})$ is represented (possible by Hall's Theorem).

The binary case

In the binary case, we must do the same thing for all other blocks B' such that $B'_{in} = B_{in}$, to avoid transforming different blocks into identical ones.

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Affine results

Note

Any polarity of $PG(2d, q)$ permutes affine d-spaces containing 0 in $AG(2d+1, q)$.

Theorem. (Clark, Jungnickel, Tonchev 2009):

Let

- α be a polarity of $PG(2d, 2)$, extended to affine d-subspaces in $AG(2d+1, 2)$, and
- $D = AG_{d+1}(2d+1, 2)$, $d \geq 2$.

Then $\alpha(D)$ is a design with the same parameters and the same 2-rank as D, but is not isomorphic to D.

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Sketch of Proof

- The block code of $AG_{d+1}(2d+1, 2)$ is a self-dual Reed-Muller code $R(d, 2d+1)$ of dimension 2^{2d}.
- The block intersection numbers of D and $\alpha(D)$ are 0 and 2^i for $1 \leq i < 2d$, and are all even.
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Note

The subdesign induced on H is isomorphic to $\alpha(PG_d(2d, 2))$. By the projective construction, its 2-rank is equal to $\mathrm{rk}_2(PG_d(2d, 2))$, but is not isomorphic to $PG_d(2d, 2)$.

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Nonisomorphic designs with geometric paramatars

- The number of nonisomorphic designs with the same parameters as $AG_{n-1}(n,q)$ or $PG_{n-1}(n,q)$, $n \geq 3$, grows **linearly** with n: **Bhat and Shrikhande (1970), Griffiths and Mavron (1972).**
- **Jungnickel (1984)**: The number of nonisomorphic designs with the same parameters as $AG_{n-1}(n,q)$ or $PG_{n-1}(n,q)$, $n \geq 3$, grows **exponentially.**
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New bounds for any $3 \leq d \leq n-1$

Jungnickel & T., 2009

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Theorem. (Clark, Jungnickel, Tonchev, 2009)

- The number of nonisomprphic **2-designs** with parameters of $AG_d(n,2)$, $d \geq 3$, grows exponentially.
- The number of nonisomprphic **3-designs** with parameters of $AG_d(n,2)$, $d \geq 3$, grows exponentially.
- The number of **resolvable 3-designs** with the parameters of $AG_d(n,2)$ grows exponentially.

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Examples

There exist at least

- 10^{228} non-isomorphic 2-$(32, 8, 35)$ designs,
- 10^{75} resolvable 2-$(32, 8, 35)$ designs,
- 10^{27} resolvable 3-$(32, 8, 7)$ designs,

all having the parameters of $AG_3(5, 2)$.

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Open Problems

Hamada's conjecture (strong form)

If D is a design having the same parameters as a geometric design G, $G = PG_d(n, q)$ or $G = AG_d(n, q)$, then

$$rank_q D \geq rank_q G,$$

with equality $rank_q D = rank_q G$ **if and only if** D is isiomorphic to G.

Note

The strong form (the "only if" part) of Hamada's conjecture is not true in general.

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Determine the spectrum of parameters n, q, d for which the strong form of Hamada's conjecture holds true.

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Assmus-Key

Hamada's conjecture is true for $PG_{n-1}(n, q)$.

Sachar

Hamada's conjecture is true for $PG_1(2, q)$, that is, for projective planes.

Note

- The Assmus-Key conjecture has been proved for $q = 2$.
- Sachar's conjecture has been verified for $q \leq 9$.

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Hamada's conjecture (weaker form)

The p-rank of $PG_d(n, p^s)$ or $AG_d(n, p^s)$ is an exact lower bound on the p-rank of all designs having the same parameters as $PG_d(n, p^s)$ or $AG_d(n, p^s)$.

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- Prove the weaker form of Hamada's conjecture.
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- **Affine case, $q > 2$:** Extend the affine construction to fields of order $q > 2$.
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Any Questions?