Galois geometries contributing to coding theory

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Outline

OUTLINE

Linear codes

- q = prime number,
- **Prime fields**: $\mathbb{F}_q = \{1, \ldots, q\} \pmod{q}$,
- **Finite fields (Galois fields):** $\mathbb{F}_q$: q prime power,
- **Linear** $[n, k, d]$**-code** C **over** $\mathbb{F}_q$ is:
  - k-dimensional subspace of $V(n, q)$,
  - *minimum distance* d = minimal number of positions in which two distinct codewords differ.

Example

Example $[5, 1, 5]$-code over $\mathbb{F}_2$;
yes $= (0, 0, 0, 0, 0)$,
no $= (1, 1, 1, 1, 1)$.
$(0, 0, 0, 0, 1)$ or $(0, 0, 0, 1, 1)$ received, most likely $(0, 0, 0, 0, 0)$ = yes transmitted.

Theorem

If in transmitted codeword at most $(d - 1)/2$ errors, it is possible to correct these errors by replacing the received n-tuple by the codeword at minimal distance.

LINEAR CODES

- **Generator matrix of $[n, k, d]$-code C**

$$G = (g_1 \cdots g_n)$$

  - $G = (k \times n)$ matrix of rank k,
  - rows of G form basis of C,
  - codeword of C = linear combination of rows of G.

Example of generator matrix

Matrix

$$G = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

generates $[7, 4, 3]$-code over $\mathbb{F}_2$.

LINEAR CODES

- **Parity check matrix H for C**
  - $(n-k) \times n$ matrix of rank $n-k$,
  - $c \in C \Leftrightarrow c \cdot H^T = \bar{0}$.

Example of parity check matrix

Matrix

$$H = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}$$

is parity check matrix for $[7, 4, 3]$-code over $\mathbb{F}_2$.

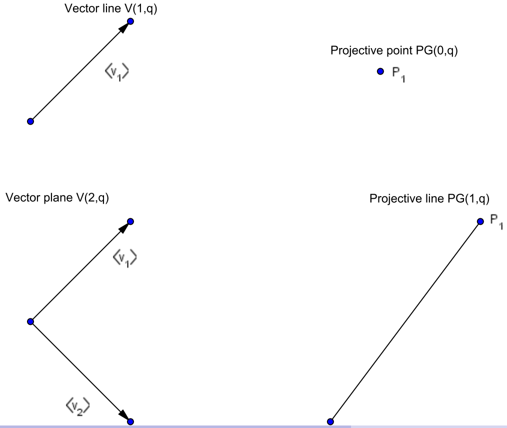
Remark

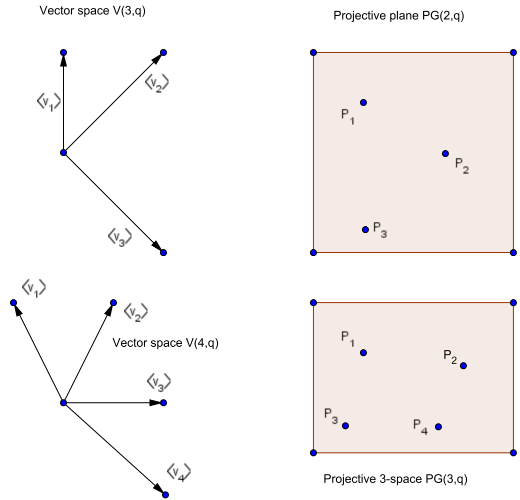
Remark: For linear $[n, k, d]$-code C, n, k, d do not change when column g_i in generator matrix

$$G = (g_1 \cdots g_n)$$

is replaced by non-zero scalar multiple.

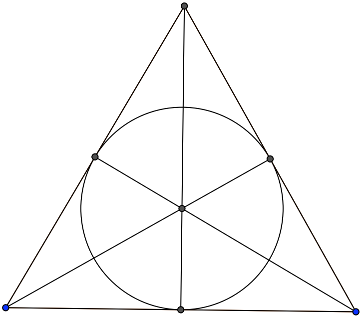
From vector space to projective space




Vector space V(3,q)
⟨v1⟩
⟨v2⟩
⟨v3⟩
Projective plane PG(2,q)
P1
P2
P3
⟨v1⟩
⟨v2⟩
Vector space V(4,q)
⟨v3⟩
⟨v4⟩
P1
P2
P3
P4
Projective 3-space PG(3,q)

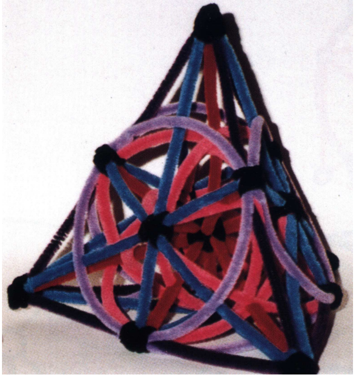
THE FANO PLANE $\mathrm{PG}(2,2)$

From $V(3,2)$ to $\mathrm{PG}(2,2)$



$\mathrm{PG}(3,2)$

From $V(4,2)$ to $\mathrm{PG}(3,2)$



OUTLINE

Griesmer bound and minihypers

Question: Given

- dimension k,
- minimal distance d,

find minimal length n of $[n, k, d]$-code over $\mathbb{F}_q$.
Result: Griesmer (lower) bound

$$n \geq \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil = g_q(k, d).$$

MINIHYPERS AND GRIESMER BOUND

Equivalence: (Hamada and Helleseth)

Griesmer (lower) bound
equivalent with
minihypers in finite projective spaces

Definition

Definition

$\{f, m; k-1, q\}$-minihyper F is:

- set of f points in $\mathrm{PG}(k-1, q)$,
- F intersects every $(k-2)$-dimensional space in at least m points.

(m-fold blocking sets with respect to the hyperplanes of $\mathrm{PG}(k-1, q)$)

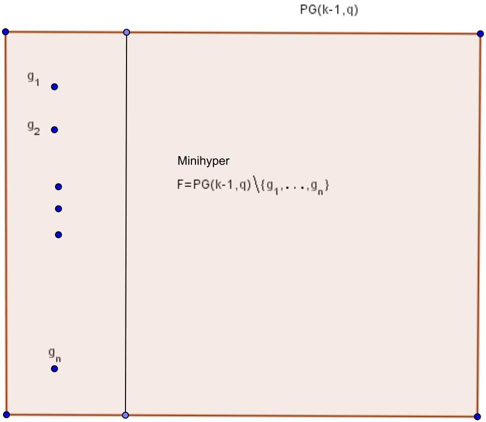
Minihypers and Griesmer bound

- Let $C = [g_q(k,d), k, d]$-code over $\mathbb{F}_q$.
- If generator matrix

$$G = (g_1 \cdots g_n),$$

minihyper = $PG(k-1,q) \setminus \{g_1, \ldots, g_n\}$.

MINIHYPERS AND GRIESMER BOUND



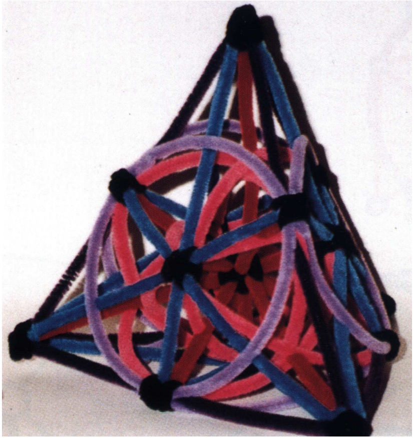
EXAMPLE

Example: Griesmer [8,4,4]-code over $\mathbb{F}_2$

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

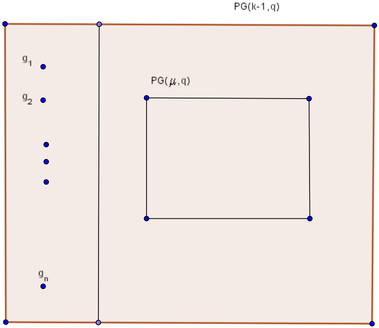
minihyper = $PG(3,2) \setminus \{\text{columns of } G\}$ = plane ($X_0 = 0$).

Corresponding minihyper



Other examples

Example 1. Subspace PG(μ, q) in PG($k-1$, q) = minihyper of $[n=(q^k-q^{\mu+1})/(q-1), k, q^{k-1}-q^{\mu}]$-code (McDonald code).



Bose-Burton theorem

Theorem (Bose-Burton)

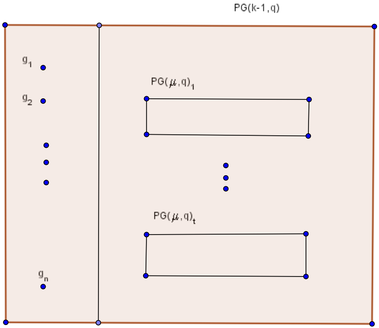
A minihyper consisting of $|PG(\mu, q)|$ points intersecting every hyperplane in at least $|PG(\mu - 1, q)|$ points is equal to a μ-dimensional space $PG(\mu, q)$.

Raj Chandra Bose

R.C. Bose and R.C. Burton, A characterization of flat spaces in a finite geometry and the uniqueness of the Hamming and the McDonald codes. *J. Combin. Theory*, 1:96-104, 1966.

OTHER EXAMPLES

Example 2. $t < q$ pairwise disjoint subspaces $\mathrm{PG}(\mu, q)_i$, $i = 1, \ldots, t$, in $\mathrm{PG}(k-1, q)$ = minihyper of $[n = (q^k - 1)/(q-1) - t(q^{\mu+1} - 1)/(q-1), k, q^{k-1} - tq^{\mu}]$-code.



CHARACTERIZATION RESULT

THEOREM (GOVAERTS AND STORME)

For q odd prime and $1 \leq t \leq (q+1)/2$,
$[n = (q^k - 1)/(q - 1) - t(q^{\mu+1} - 1)/(q - 1), k, q^{k-1} - tq^{\mu}]$-code
C: minihyper is union of t pairwise disjoint $PG(\mu, q)$.

Outline

Definition

Definition

Let C be linear $[n, k, d]$-code over $\mathbb{F}_q$. The *covering radius* of C is smallest integer R such that every n-tuple in $\mathbb{F}_q^n$ differs in at most R positions from some codeword in C.

Theorem

Let C be linear $[n, k, d]$-code over $\mathbb{F}_q$ with parity check matrix

$$H = (h_1 \cdots h_n).$$

Then covering radius of C is equal to R if and only if every $(n-k)$-tuple over $\mathbb{F}_q$ can be written as linear combination of at most R columns of H.

DEFINITION

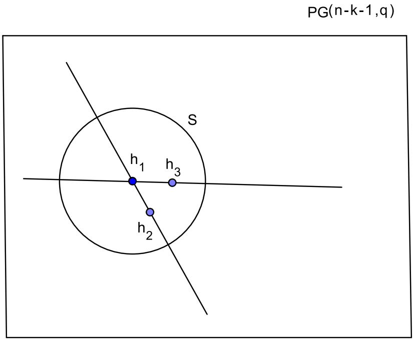
DEFINITION

Let S be subset of $\mathrm{PG}(N, q)$. The set S is called ρ-*saturating* when every point P from $\mathrm{PG}(N, q)$ can be written as linear combination of at most $\rho + 1$ points of S.

Covering radius ρ for linear $[n, k, d]$-code equivalent with
$(\rho - 1)$-*saturating set in PG*$(n - k - 1, q)$

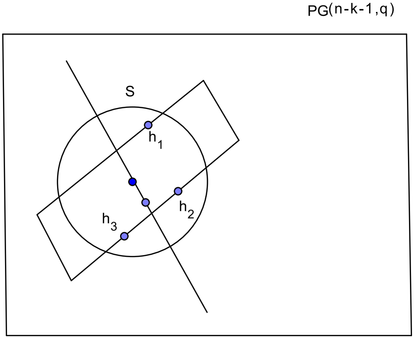
1-SATURATING SETS

$$H = (h_1 \cdots h_n)$$

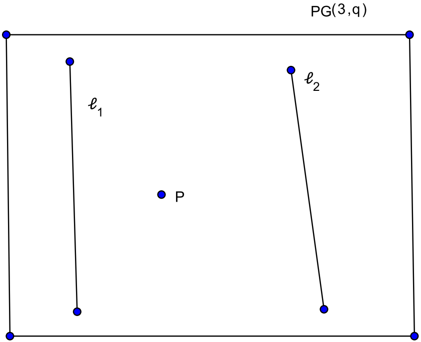


2-SATURATING SETS

$$H = (h_1 \cdots h_n)$$



1-Saturating set in $\mathrm{PG}(3, q)$ of size $2q + 2$



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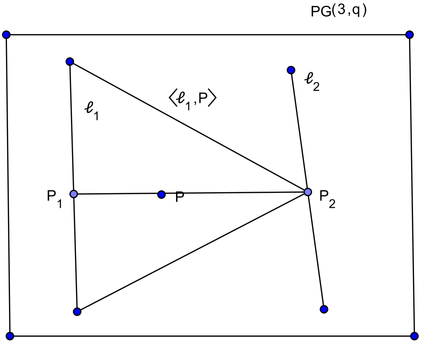


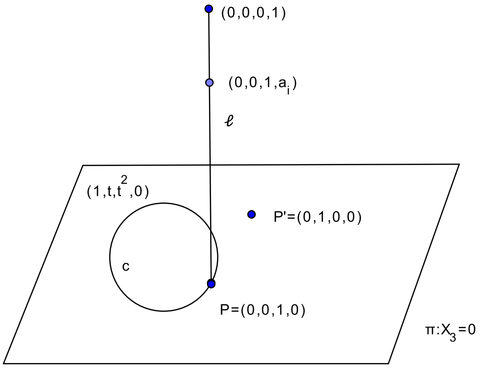
Example of Östergård and Davydov

Let $\mathbb{F}_q = \{a_1 = 0, a_2, \ldots, a_q\}$.

$$H_1 = \left[\begin{array}{ccc|c|cccc} 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ a_1 & \cdots & a_q & 1 & 0 & 0 & \cdots & 0 \\ a_1^2 & \cdots & a_q^2 & 0 & 0 & 1 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & 1 & a_2 & \cdots & a_q \end{array}\right]$$

Columns of H_1 define 1-saturating set of size $2q + 1$ in $\mathrm{PG}(3, q)$.

Example of Östergård and Davydov

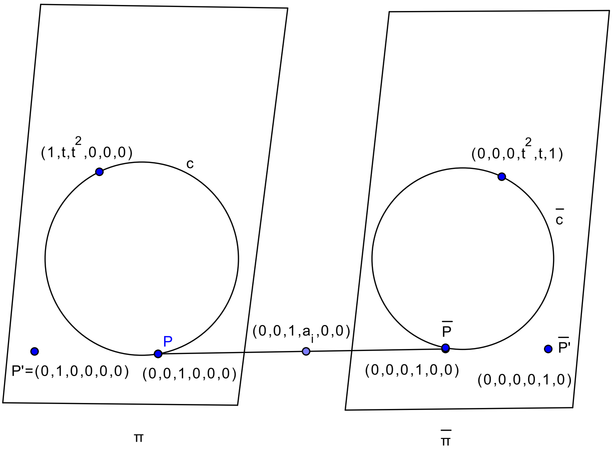


Example of Östergård and Davydov

$$H_2 = \left[\begin{array}{ccc|c|ccc|ccc|c} 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ a_1 & \cdots & a_q & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ a_1^2 & \cdots & a_q^2 & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & a_2 & \cdots & a_q & a_1^2 & \cdots & a_q^2 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_1 & \cdots & a_q & 1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 \end{array}\right],$$

Columns of H_2 define 2-saturating set of size $3q+1$ in $\mathrm{PG}(5,q)$.

Example of Östergård and Davydov



OUTLINE

Linear MDS codes and arcs

Question:
Given

- length n,
- dimension k,

find maximal value of d.
Result: Singleton (upper) bound

$$d \leq n - k + 1.$$

Notation: MDS code = $[n, k, n-k+1]$-code.

ARCS

Equivalence:

Singleton (upper) bound (MDS codes)
equivalent with
***Arcs in finite projective spaces* (Segre)**

Definition

Definition

n-Arc in PG$(k-1, q)$: set of n points, every k linearly independent.

Example:

1. n-arc in PG$(2, q)$: n points, no three collinear.
2. Conic $X_1^2 = X_0X_2$

$$\{(1, t, t^2) || t \in \mathbb{F}_q\} \cup \{(0, 0, 1)\}$$

Normal rational curve

Classical example of arc:

$$\{(1, t, \ldots, t^{k-1}) || t \in \mathbb{F}_q\} \cup \{(0, \ldots, 0, 1)\}$$

defines $[q+1, k, d = q+2-k]$-GDRS (**Generalized Doubly-Extended Reed-Solomon**) code with generator matrix

$$G = \begin{pmatrix} 1 & \cdots & 1 & 0 \\ t_1 & \cdots & t_q & 0 \\ \vdots & \vdots & \vdots & \vdots \\ t_1^{k-2} & \cdots & t_q^{k-2} & 0 \\ t_1^{k-1} & \cdots & t_q^{k-1} & 1 \end{pmatrix}$$

Characterization result

Theorem (Segre, Thas)

For

- *q odd prime power,*
- $2 \leq k < \sqrt{q}/4$,

$[n = q + 1, k, d = q + 2 - k]$*-MDS code is GDRS.*

BALL RESULT

THEOREM (BALL)

For q odd prime, $n \leq q+1$ for every $[n, k, n-k+1]$-MDS code.

Technique: Polynomial techniques

OUTLINE

Well-known extendability result

Theorem

Every linear binary $[n, k, d]$-code C, d odd, is extendable to linear binary $[n+1, k, d+1]$-code.

Hill-Lizak result

Theorem (Hill and Lizak)

Let C be linear $[n, k, d]$-code over $\mathbb{F}_q$, with $\gcd(d, q) = 1$ and with all weights congruent to 0 or $d \pmod q$. Then C can be extended to $[n+1, k, d+1]$-code $\hat{C}$ all of whose weights are congruent to 0 or $d+1 \pmod q$.

Proof: Subcode of all codewords of weight congruent to 0 $\pmod q$ is linear subcode C_0 of dimension $k-1$. If G_0 defines C_0 and

$$G = \left(\frac{x}{G_0} \right),$$

then

HILL-LIZAK RESULT

$$\hat{G} = \left(\begin{array}{c|c} x & 1 \\ \hline & 0 \\ G_0 & \vdots \\ & 0 \end{array} \right)$$

defines $\hat{C}$. $\square$

Geometrical counterpart of Landjev

Definition

Multiset K in $\mathrm{PG}(k-1, q)$ is $(n, w; k-1, q)$-*multiarc* or $(n, w; k-1, q)$-*arc* if

1. sum of all weights of points of K is n,
2. hyperplane H of $\mathrm{PG}(k-1, q)$ contains at most w (weighted) points of K and some hyperplane H_0 contains w (weighted) points of K.

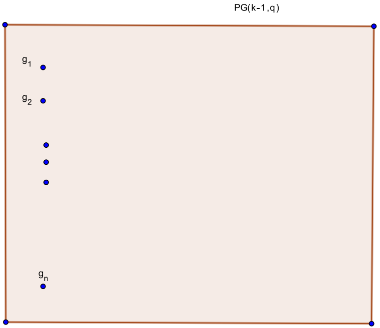
LINEAR CODES AND MULTIARCS

- Let $C = [n, k, d]$-code over $\mathbb{F}_q$.
- If generator matrix

$$G = (g_1 \cdots g_n),$$

then $\{g_1, \ldots, g_n\} = (n, w = n - d; k - 1, q)$-multiarc.

Linear codes and multiarcs



GEOMETRICAL COUNTERPART OF LANDJEV

- C linear $[n, k, d]$-code over $\mathbb{F}_q$, $\gcd(d, q) = 1$ and with all weights congruent to 0 or $d \pmod{q}$. Then C can be extended to $[n+1, k, d+1]$-code all of whose weights are congruent to 0 or $d+1 \pmod{q}$.
- K =$(n, w; k-1, q)$-multiarc with $\gcd(n-w, q) = 1$ and intersection size of K with all hyperplanes congruent to n or $w \pmod{q}$. Then K can be extended to $(n+1, w; k-1, q)$-multiarc.

Geometrical counterpart of Landjev

Proof: Hyperplanes H containing $n \pmod q$ points of K form dual blocking set $\tilde{B}$ with respect to codimension 2 subspaces of $\mathrm{PG}(k-1,q)$. Also

$$\tilde{B} = \frac{q^{k-1}-1}{q-1}.$$

By dual of Bose-Burton, $\tilde{B}$ consists of all hyperplanes through particular point P.
This point P extends K to $(n+1, w; k-1, q)$-multiarc. $\square$

Blocking sets in PG$(2, q)$

Definition

Blocking set B in PG$(2, q)$: intersects every line in at least one point.

Trivial example: Line.

Definition

Non-trivial blocking set in PG$(2, q)$: contains no line.

Blocking sets in $\mathrm{PG}(2, q)$

$q + r(q) + 1$ = size of smallest non-trivial blocking set in $\mathrm{PG}(2, q)$.

- (Blokhuis) $r(q) = (q + 1)/2$ for $q > 2$ prime,
- (Bruen) $r(q) = \sqrt{q} + 1$ for q square,
- (Blokhuis) $r(q) = q^{2/3} + 1$ for q cube (non-square) power.

Improved results

Theorem (Landjev and Rousseva)

Let $\mathcal{K}$ be $(n, w; k-1, q)$-arc, $q = p^s$, with spectrum $(a_i)_{i \geq 0}$. Let $w \not\equiv n \pmod{q}$ and

$$\sum_{i \not\equiv w \pmod{q}} a_i < q^{k-2} + q^{k-3} + \cdots + 1 + q^{k-3} \cdot r(q), \quad (1)$$

where $q + r(q) + 1$ is minimal size of non-trivial blocking set of $PG(2, q)$. Then $\mathcal{K}$ is extendable to $(n+1, w; k-1, q)$-arc.

IMPROVED RESULTS

THEOREM

Let C be non-extendable $[n, k, d]$-code over $\mathbb{F}_q$, $q = p^s$, with $\gcd(d, q) = 1$. If $(A_i)_{i\geq 0}$ is spectrum of C, then $\sum_{i\not\equiv 0,d \pmod q} A_i \geq q^{k-3} \cdot r(q)$, where $q + r(q) + 1$ is minimal size of non-trivial blocking set of $PG(2, q)$.

Thank you very much for your attention!