Finite projective spaces

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Outline

1. Finite fields
   - Prime fields
2. Projective plane PG$(2, q)$
   - Points and lines
   - Coordinates
3. Projective space PG$(3, q)$
   - Points, lines, and planes
   - Equations
   - PG$(3, 2)$
4. Blocking sets

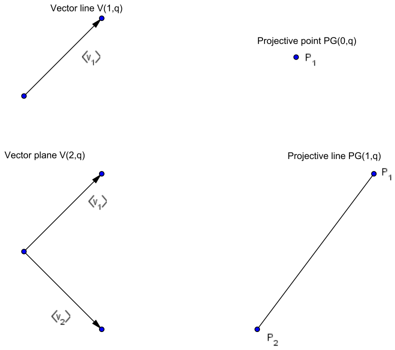
Outline

Finite fields

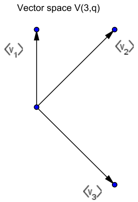
- q = prime number.
  - **Prime fields** $\mathbb{F}_q = \{0, 1, \ldots, q-1\} \pmod q$.
  - Binary field $\mathbb{F}_2 = \{0, 1\}$.
  - Ternary field $\mathbb{F}_3 = \{0, 1, 2\} = \{-1, 0, 1\}$.
- **Finite fields** $\mathbb{F}_q$: q prime power.

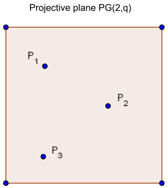
Outline

FROM $V(3, q)$ TO PG$(2, q)$



FROM $V(3, q)$ TO PG$(2, q)$





Points and lines

Theorem

PG$(2, q)$ has $q^2 + q + 1$ points and $q^2 + q + 1$ lines.

Proof:

- $(q^3 - 1)/(q - 1) = q^2 + q + 1$ vector lines in $V(3, q)$.
- Vector plane in $V(3, q)$: $a_0X_0 + a_1X_1 + a_2X_2 = 0$.
  $(q^3 - 1)/(q - 1) = q^2 + q + 1$ vector planes in $V(3, q)$. □

Points on lines

Theorem

(1) Two points in $PG(2, q)$ belong to unique line of $PG(2, q)$.
(2) Two lines in $PG(2, q)$ intersect in unique point.

Proof:

- Two vector lines in $V(3, q)$ define unique vector plane in $V(3, q)$.
- Two vector planes in $V(3, q)$ intersect in unique vector line in $V(3, q)$.

□

Points on lines

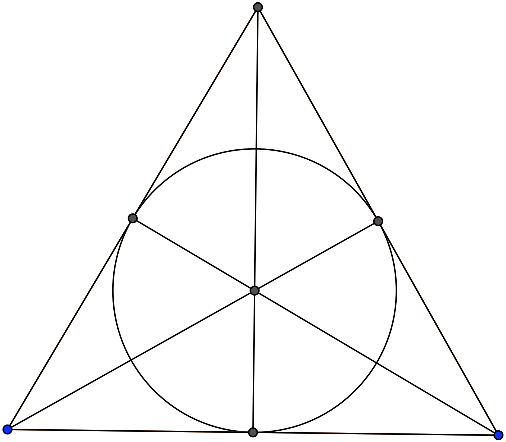
Theorem

(1) Line of $PG(2, q)$ *has* $q + 1$ *points.*
(2) Point of $PG(2, q)$ *lies on* $q + 1$ *lines of* $PG(2, q)$*.*

Proof:

- Vector plane of $V(3, q)$ has $q^2 - 1$ non-zero vectors; each vector line has $q - 1$ non-zero vectors, so vector plane of $V(3, q)$ has $(q^2 - 1)/(q - 1) = q + 1$ vector lines.
- Take vector line $\langle(1, 0, 0)\rangle$. This lies in vector planes $a_1X_1 + a_2X_2 = 0$. Up to non-zero scalar multiple of $(a_1, a_2) \neq (0, 0)$, these equations define $(q^2 - 1)/(q - 1) = q + 1$ vector planes of $V(3, q)$. □

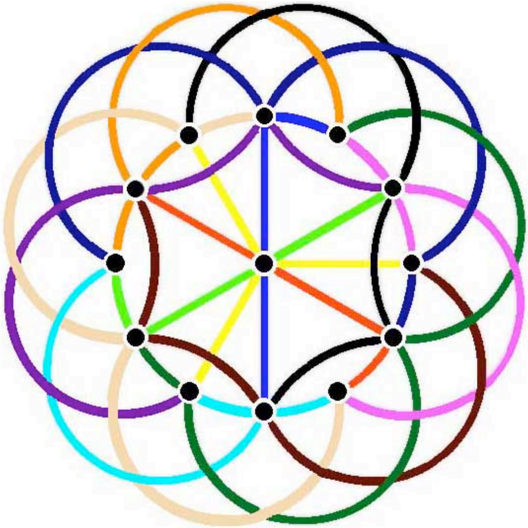
THE FANO PLANE PG(2, 2)



PROPERTIES OF FANO PLANE

- PG(2, 2) has 7 points:
  $\langle(a_0, a_1, a_2)\rangle = \{(0, 0, 0), (a_0, a_1, a_2)\} \equiv (a_0, a_1, a_2)$.
- PG(2, 2) has 7 lines: $a_0X_0 + a_1X_1 + a_2X_2 = 0$.

THE PLANE PG$(2, 3)$



Properties of PG(2, 3)

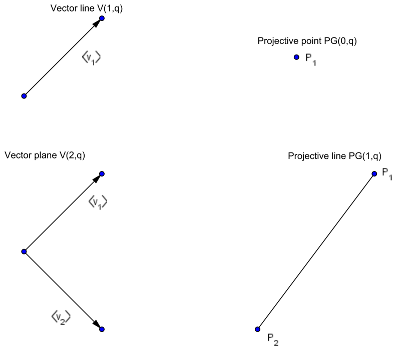
- PG(2, 3) has 13 points.
  Vector line
  $\langle (a_0, a_1, a_2) \rangle = \{(0,0,0), (a_0, a_1, a_2), 2 \cdot (a_0, a_1, a_2)\}$.
- PG(2, 3) has 13 lines: $a_0 X_0 + a_1 X_1 + a_2 X_2 = 0$.

Normalized coordinates

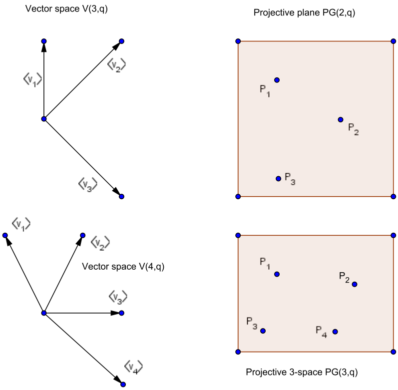
- Projective point = vector line $\langle(a_0, a_1, a_2)\rangle$.
- Select leftmost non-zero coordinate equal to one.
- **Example:** In PG$(2, 3)$,
  Point $(2, 2, 0) \equiv (1, 1, 0)$.

OUTLINE

FROM $V(4, q)$ TO PG$(3, q)$



FROM $V(4, q)$ TO PG$(3, q)$



Points and planes

Theorem

$PG(3, q)$ has $q^3 + q^2 + q + 1$ points and $q^3 + q^2 + q + 1$ planes.

Proof:

- $(q^4 - 1)/(q - 1) = q^3 + q^2 + q + 1$ vector lines in $V(4, q)$.
- 3-dimensional vector space in $V(4, q)$:
  $a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 = 0$.
  $(q^4 - 1)/(q - 1) = q^3 + q^2 + q + 1$ 3-dimensional vector spaces in $V(4, q)$. □

Lines in PG$(3, q)$

Theorem

PG$(3, q)$ has $(q^2 + 1)(q^2 + q + 1)$ lines.

Proof: 2 points define a line, containing $q + 1$ points. So

$$\frac{(q^3 + q^2 + q + 1)(q^3 + q^2 + q)}{(q + 1)q} = (q^2 + 1)(q^2 + q + 1)$$

lines in PG$(3, q)$. □

Points on lines

Theorem

(1) Two points in PG$(3, q)$ belong to unique line of PG$(3, q)$.
(2) Two lines in PG$(3, q)$ intersect in zero or one points.

Proof:

- Two vector lines in $V(4, q)$ define unique vector plane in $V(4, q)$.
- Two vector planes in $V(4, q)$ intersect in unique vector line in $V(4, q)$, or only in zero vector.

□

Points on lines

Theorem

(1) Two planes in PG$(3, q)$ intersect in unique line of PG$(3, q)$.
(2) A line and a plane in PG$(3, q)$ intersect in one point if the line is not contained in this plane.

Proof:

- Two 3-dimensional vector spaces in $V(4, q)$ intersect in unique vector plane in $V(4, q)$.
- Vector plane in $V(4, q)$ and 3-dimensional vector space in $V(4, q)$ intersect in unique vector line in $V(4, q)$, if vector plane is not contained in 3-dimensional vector space.

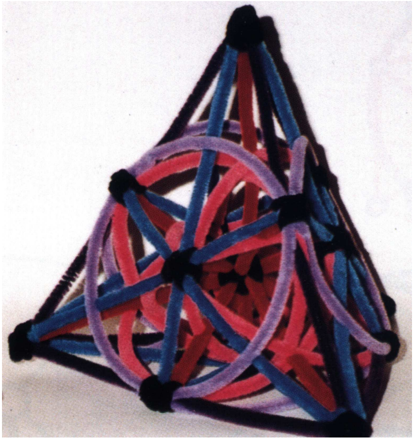
EQUATIONS FOR LINES AND PLANES IN PG(3, q)

- Plane: $a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 = 0$.
- Line:

$$\begin{cases} a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 & = & 0 \\ b_0X_0 + b_1X_1 + b_2X_2 + b_3X_3 & = & 0, \end{cases}$$

where $(a_0, a_1, a_2, a_3), (b_0, b_1, b_2, b_3) \neq (0, 0, 0, 0)$ and where $(a_0, a_1, a_2, a_3) \neq \rho(b_0, b_1, b_2, b_3)$.

PG$(3, 2)$



FROM $V(n+1, q)$ TO PG(n, q)

1. From $V(1, q)$ to PG$(0, q)$ (projective point),
2. From $V(2, q)$ to PG$(1, q)$ (projective line),
3. ...
4. From $V(i+1, q)$ to PG(i, q) (i-dimensional projective subspace),
5. ...
6. From $V(n, q)$ to PG$(n-1, q)$ ($(n-1)$-dimensional subspace = hyperplane),
7. From $V(n+1, q)$ to PG(n, q) (n-dimensional space).

Outline

Definition and example

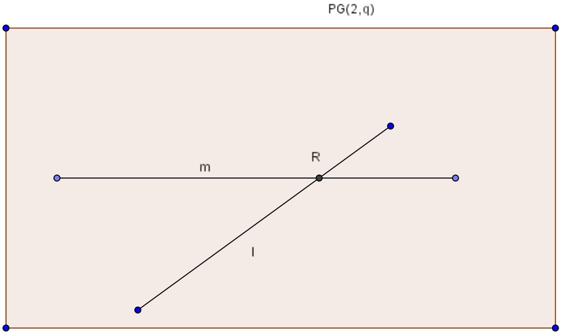
Definition

Blocking set B in PG(2, q) is set of points, intersecting every line in at least one point.

Example

Line L in PG(2, q).

EXAMPLE



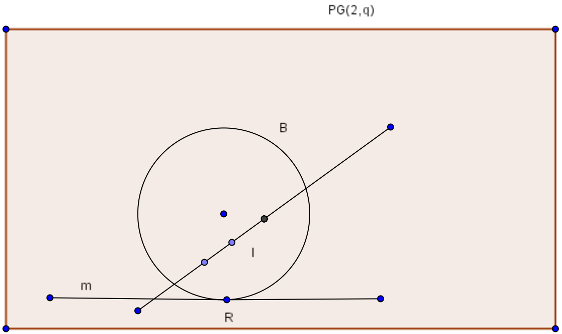
DEFINITION

DEFINITION

(1) Point r of blocking set B in PG$(2, q)$ is *essential* if $B \setminus \{r\}$ is no longer blocking set.
(2) *Tangent line* L to blocking set B in PG$(2, q)$ is line for which $|L \cap B| = 1$.

THEOREM

Point r of blocking set B is essential if and only if r belongs to tangent line L to B.


PG(2,q)
B
l
m
R

Minimal blocking sets

Definition

Blocking set B is *minimal* if and only if all of its points are essential.

Example

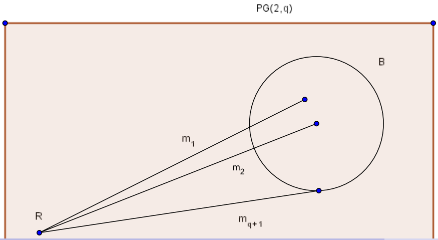
Line L of PG$(2, q)$ is minimal blocking set B of size $q + 1$.

Bose-Burton theorem

Theorem

For every blocking set B in PG$(2, q)$, $|B| \geq q + 1$ and $|B| = q + 1$ if and only if B is equal to line L.

Proof: (1) Let $r \notin B$.



(2) Let $|B| = q + 1$.
Part (1) shows that line L not contained in B only contains one point of B.
So, let $r_1, r_2 \in B$, then line $r_1 r_2$ contains at least 2 points of B, then $r_1 r_2 \subseteq B$. □

Lower bound on size of non-trivial blocking set in PG$(2, q)$

Definition

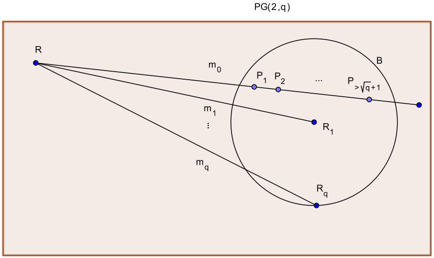
Non-trivial blocking set B in PG$(2, q)$ does not contain a line.

Theorem

For non-trivial blocking set B in PG$(2, q)$, $|B| \geq q + \sqrt{q} + 1$.

Lower bound on size of non-trivial blocking set in PG$(2, q)$

Proof: (1) Suppose some line L contains more than $\sqrt{q}+1$ points of B, then $|B| > q + \sqrt{q} + 1$.



Lower bound on size of non-trivial blocking set in PG$(2, q)$

(2) From now on, assume every line contains at most $\sqrt{q} + 1$ points of B.
Let τ_i be number of i-secants to B; let n be largest number of points of B on line of PG$(2, q)$. Then

Lower bound on size of non-trivial blocking set in PG(2, q)

$$\sum_{i=1}^{n} \tau_i = q^2 + q + 1, \qquad (1)$$

$$\sum_{i=1}^{n} i\tau_i = |B|(q + 1), \qquad (2)$$

$$\sum_{i=2}^{n} i(i-1)\tau_i = |B|(|B| - 1). \qquad (3)$$

Lower bound on size of non-trivial blocking set in PG$(2, q)$

Meaning of (1), (2), and (3):

(1) number of lines in PG$(2, q)$,

(2) count pairs (P, ℓ), with $P \in B$, line ℓ, and $P \in \ell$,

(3) count triples (P, P', ℓ), with $P, P' \in B$, $P \neq P'$, line ℓ, and $P, P' \in \ell$.

Lower bound on size of non-trivial blocking set in PG$(2, q)$

Since $1 \leq |L \cap B| \leq n \leq \sqrt{q} + 1$, for all lines L,

$$\begin{aligned}
\sum_{i=1}^{n} (i-1)(i-\sqrt{q}-1)\tau_i &\leq 0, \\
\sum_{i=1}^{n} i(i-1)\tau_i - (\sqrt{q}+1)\sum_{i=1}^{n} i\tau_i + (\sqrt{q}+1)\sum_{i=1}^{n}\tau_i &\leq 0, \\
|B|(|B|-1) - (\sqrt{q}+1)|B|(q+1) + \\
(\sqrt{q}+1)(q^2+q+1) &\leq 0, \\
(|B| - (q+\sqrt{q}+1))(|B| - (q\sqrt{q}+1)) &\leq 0.
\end{aligned}$$

So $|B| \geq q + \sqrt{q} + 1$.

General blocking sets

Definition

Blocking set B in PG(n, q) with respect to k-subspaces is set of points, intersecting every k-subspace in at least one point.

Example

$(n - k)$-dimensional subspace PG$(n - k, q)$ in PG(n, q).

Bose-Burton theorem

Theorem

For every blocking set B in $PG(n, q)$, with respect to the k-subspaces, $|B| \geq PG(n-k, q)$ and $|B| = |PG(n-k, q)|$ if and only if B is equal to $(n-k)$-dimensional subspace $PG(n-k, q)$.

Thank you very much for your attention!