Quantum jump codes and related combinatorial designs

Masakazu JIMBO [a,1], Keisuke SHIROMOTO [b]

[a] *Graduate School of Information Science, Nagoya University, Nagoya, Japan*

[b] *Department of Mathematics and Engineering, Kumamoto University, Kumamoto, Japan*

Abstract. Quantum jump codes were introduced by Alber et al. (2001). Quantum jump codes have a close connection with combinatorial designs called t-SEED (t-spontaneaus emission error design). In this paper, we give a brief survey of a quantum jump code together with some new results. Firstly, fundamental properties of a t-error correcting quantum jump code are described. Secondly, a few examples of jump codes are given and an upperbound of dimension of a jump code with a fixed length and given error correcting ability is shown. Moreover, a relation between an t-SEED and a jump code is discussed and various constructions of t-SEEDs are given.

Keywords. quantum jump code, t-SEED, large set

Introduction

Quantum error correcting codes have been studied by many authors [9,11,14,29,30] motivated by the pioneering work by Shor [28]. Among them, Alber *et al.* [1] introduced quantum jump codes which correct errors caused by quantum jumps. Quantum jump codes have a close connection with combinatorial designs called t-SEED (t-spontaneaus emission error design).

In this paper, we give a brief survey of a quantum jump code together with some new results. In Sections 1 and 2, a breif introduction to a quantum jump code is given. In Section 3, a few examples of jump codes are shown and in Section 4, an upperbound of dimension of a jump code with a fixed length and given error correcting ability is explained. In Section 5, a non-existence result for a special parameter is shown. Moreover, in Section 6 a connection between a t-SEED and a jump code is discussed. Finally, in Section 7, various constructions of t-SEEDs are given.

1. Quantum codes

We begin with the introduction of quantum error correcting codes.

[1] Corresponding Author: Masakazu Jimbo, E-mail: jimbo@is.nagoya-u.ac.jp

1.1. Quantum state

A quantum state for a single particle can be represented by a vector in a finite dimensional Hilbert space $\mathcal{H}$, that is, a vector space with inner product. In this paper we set $\mathcal{H} = \mathbb{C}^2$, where $\mathbb{C}$ is the set of complex numbers. A quantum state of a single quantum system like a photon is represented by $|\varphi\rangle$, called a *ket vector*, which is a 2-dimesional vector in $\mathcal{H} = \mathbb{C}^2$. The unit of the information amount for a single quantum system is called a *qubit*. In particular, $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are called *pure states* and any state $|\varphi\rangle$ is a linear combination (*superposition*) of these two pure states, that is, $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$ for $\alpha, \beta \in \mathbb{C}$. We define a *bra vector* $\langle\varphi| = |\varphi\rangle^\dagger$, where $|\varphi\rangle^\dagger$ is the tanspose of the complex conjugate of $|\varphi\rangle$. Then the inner product of $|\varphi\rangle$ and $|\psi\rangle$ is written by the notation $\langle\varphi|\psi\rangle$ and the size of a state vector $|\varphi\rangle$ is written by $\sqrt{\langle\varphi|\varphi\rangle}$.

In the field of quantum information, any state $|\varphi\rangle$ and its scalar multiple $\alpha|\varphi\rangle$ $(\alpha \neq 0)$ are identified as the same quantum state. Hence, without loss of generality, we assume $\langle\varphi|\varphi\rangle = 1$.

A joint state of n-qubits is of the form

$$|\varphi\rangle = |\varphi_1\varphi_2 \cdots \varphi_n\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes \cdots \otimes |\varphi_n\rangle,$$

where $\otimes$ is the tensor product. In this case, $|\varphi\rangle$ is a 2^n-dimensional vector in $\mathcal{H}^{\otimes n} = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}$.

Let $\mathcal{F} = \{0, 1\}$ and $\mathcal{F}^n = \mathcal{F} \times \cdots \times \mathcal{F}$. Then for any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{F}^n$, $|\boldsymbol{x}\rangle = |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$ are pure states for n-qubits and these 2^n vectors in $\{|\boldsymbol{x}\rangle : \boldsymbol{x} \in \mathcal{F}^n\}$ form an orthonormal basis of $\mathcal{H}^{\otimes n}$. Any n-qubit state can be represented by $|\varphi\rangle = \sum_{\boldsymbol{x} \in \mathcal{F}^n} \alpha_{\boldsymbol{x}} |\boldsymbol{x}\rangle$.

1.2. State Transition

For any quantum state $|\varphi\rangle \in \mathcal{H}^{\otimes n}$, a state transition can be represented by a linear operator. In a quantum computation or quantum data transmission, information is stored as a quantum state of an n-qubit system. Quantum computation can be pursued by applying suitable unitary operators. However, in these computation or data transmission system, we can not avoid the occurence of errors or noises caused by the interaction with environment. Because of the noise, the information stored in a quantum system may include some error. Errors or noises are also considered as operators. Typical unitary operators for a single qubit are

$$\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

where $i = \sqrt{-1}$.

In order to correct such errors, we need to apply (inverse) unitary operators. But unlike to the classical data strage, we can not observe the quantum state of the system. Hence, we need to correct the quantum state by utilizing some partial information like an eigenvalue of a measurement, or by observing changes of outside of the system.

1.3. Quantum error correcting codes

Let $\mathcal{C}$ be a subspace of $\mathcal{H}^{\otimes n}$ and $\mathcal{E}$ be a set of error operators including the identity operator. We assume that only the errors in $\mathcal{E}$ occur in a quantum system. $\mathcal{C}$ is called an *$\mathcal{E}$-error correcting quantum codes* if, for any $|c\rangle \in \mathcal{C}$ and $E \in \mathcal{E}$, we can recover the original state $|c\rangle$ utilizing a partial information of $E|c\rangle$ obatined by measurement without knowing the original state $|c\rangle$.

For an $\mathcal{E}$-error correcting quantum code $\mathcal{C}$, the following theorem is known.

Theorem 1 (Knill and Laflamme [23]) *A subspace $\mathcal{C} \le \mathcal{H}^{\otimes n}$ with orthonormal basis $\{|c_i\rangle : i = 1, \ldots, m\}$ is a quantum $\mathcal{E}$-error correcting code if and only if the following holds:*

$$\langle c_i|E_1^\dagger E_2|c_j\rangle = \delta_{ij}\kappa_{E_1,E_2}, \quad \text{for any } i, j, \text{ and } E_1, E_2 \in \mathcal{E}, \tag{1}$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and κ_{E_1,E_2} is a constant depeding only on E_1 and E_2.

Example 2 *Let $n = 2$ and $\mathcal{E} = \{I \otimes I, E = I \otimes \sigma_X\}$. Then, $\mathcal{C} = \langle |00\rangle, |11\rangle\rangle$ is an $\mathcal{E}$-error correcting quantum code, since $E|00\rangle = |01\rangle$, $E|11\rangle = |10\rangle$ implies $\langle 00|E^\dagger E|11\rangle = 0$, $\langle 00|E|11\rangle = 0$, $\langle 00|E|00\rangle = \langle 11|E|11\rangle = 0$, $\langle 00|E^\dagger E|00\rangle = \langle 11|E^\dagger E|11\rangle = 1$, which satisfies the condition (1).*

2. A quantum jump code

2.1. A decay operator and a jump operator

In this paper, we treat errors caused by spontaneous emission. Quantum state is changed according as the spontaneous emission by the loss of energy. In this case, there are two kinds of errors, that is, quantum decay and quantum jump. A *quantum decay operator* is represented by

$$D(t) = e^{-\frac{\kappa t}{2}|1\rangle\langle 1|} = e^{-\frac{\kappa t}{2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e^{-\frac{\kappa t}{2}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

where t is a time variable and κ is a decay rate. Then, for $x = 0, 1$,

$$D(t)|x\rangle = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + e^{-\frac{\kappa t}{2}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) |x\rangle = e^{-x\frac{\kappa t}{2}}|x\rangle. \tag{2}$$

holds.

Assume that spontaneous decay occurs to each qubit with the same decay rate. Then the decay operator for n-qubit quantum state is defined by $D_V(t) = D(t)\otimes\cdots\otimes D(t) = D(t)^{\otimes n}$. For any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{F}^n$, we have

$$D_V(t)|\boldsymbol{x}\rangle = \bigotimes_{i=1}^{n} D(t)|x_i\rangle = e^{-\mathrm{wt}(\boldsymbol{x})\frac{\kappa t}{2}}|\boldsymbol{x}\rangle, \tag{3}$$

where wt($\boldsymbol{x}$) is the Hamming weight of $\boldsymbol{x}$, that is, the number of nonzero elements in $\boldsymbol{x}$.

On the other hand quantum jump is defined as follows: Let

$$A = |0\rangle\langle 1| = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

then the *quantum jump operator* for a single qubit is defined by

$$J|\phi\rangle = \begin{cases} A|\phi\rangle, & \text{if } \langle\phi|A^\dagger A|\phi\rangle \neq 0, \\ |\phi\rangle & \text{if } \langle\phi|A^\dagger A|\phi\rangle = 0. \end{cases} \tag{4}$$

Thus, we have

$$J(\alpha|0\rangle + \beta|1\rangle) = \begin{cases} \beta|0\rangle, & \text{if } \beta \neq 0, \\ |0\rangle, & \text{if } \beta = 0. \end{cases}$$

Remark: In Alber *et al.* [2], a jump operator is defined by

$$J|\varphi\rangle = \frac{1}{\sqrt{\langle\varphi|A^\dagger A|\varphi\rangle}} A|\varphi\rangle.$$

Hence, in the case of $|\varphi\rangle = |0\rangle$, by setting $|\varphi\rangle = \varepsilon_0|0\rangle + \varepsilon_1|1\rangle$, $(|\varepsilon_0|^2 + |\varepsilon_1|^2 = 1)$, we should consider it as

$$J|0\rangle = \lim_{\varepsilon_1 \longrightarrow 0} \frac{1}{\sqrt{\langle\varphi|A^\dagger A|\varphi\rangle}} A|\varphi\rangle = |0\rangle.$$

In this paper, we ignore the normalizing denominator $\sqrt{\langle\varphi|A^\dagger A|\varphi\rangle}$ and instead we defined that any state vector $|\varphi\rangle$ is identified with its scalar multiple in Subsection 1.1.

In the case of n-qubit system, a jump operator at the i-th position is defined by

$$J_i = I \otimes I \otimes \cdots \otimes I \otimes J \otimes I \cdots \otimes I.$$

Let $V = \{1, 2, \ldots, n\}$ and $\mathcal{L}_U$ be the set of lists of the elements in $U \subset V$. If jump error operators $J_{i_1}, \ldots, J_{i_{s-1}}, J_{i_s}$ are applied in turn to a quantum state $|c\rangle$, such multiple jump is represented by $J_E = J_{i_s} J_{i_{s-1}} \cdots J_{i_1}$ for $E = (i_1, \cdots, i_s) \in \mathcal{L}_E$. In general, jump operators $J_{i_1}, \ldots, J_{i_{s-1}}, J_{i_s}$ are not commutative. For example,

$J_2 J_1(|101\rangle + |010\rangle) = |001\rangle$, whereas $J_1 J_2(|101\rangle + |010\rangle) = |000\rangle$. However, for a state $|c\rangle$, by deleting jump operator J_{i_j}'s which do not change the state $J_{i_{j-1}} \cdots J_{i_1}|c\rangle$ we can get a subsequence of operators $J_{E_c} = J_{i_{j_r}} \cdots J_{i_{j_1}}$, where $E_c = (i_{j_1}, \cdots, i_{j_r}) \subset E$. Hence $J_E|c\rangle = J_{E_c}|c\rangle$ holds for the state $|c\rangle = \sum_{\boldsymbol{x} \in \mathcal{F}^n} \alpha_{\boldsymbol{x}}|\boldsymbol{x}\rangle$ and there are $\boldsymbol{x}$'s such that $\alpha_{\boldsymbol{x}} \neq 0$ and $\mathrm{supp}(\boldsymbol{x}) \supset E_c$ hold, where $\mathrm{supp}(\boldsymbol{x}) = \{i : x_i \neq 0\}$ for $\boldsymbol{x} = (x_1, \ldots, x_n)$. Moreover, the operators in J_{E_c} are commutative when it is applied to $|c\rangle$. Hence, for a multiple jump operator J_E and a state $|c\rangle$, we have only to consider multiple jump operators which are commutative with respect to $|c\rangle$. Now, for a subset E of V, when the jumps at positions in E are commutative for $|c\rangle$, we denote it by

$$J_E = \bigotimes_{i=1}^{n} A_i, \quad A_i = \begin{cases} I & \text{if } i \notin E, \\ J & \text{if } i \in E. \end{cases}$$

The position where a quantum jump occured can be detected by the continuous monitoring of photodetector since a photon is radiated when a quantum jump occured at a qubit (see Figure 1). Hence, we assume that the positions where quantum jumps occur are known (see, Alber *et al.* [2]).

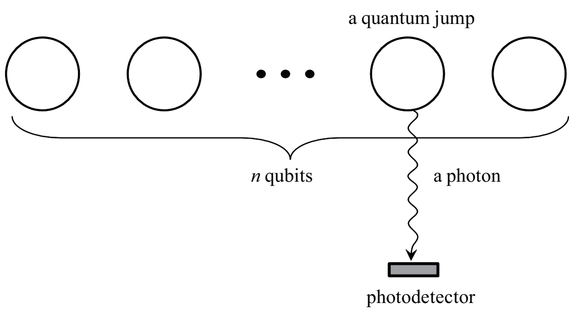

Figure 1. A quantum jump and photodetector

In general, a decay and jump process is written as

$$D_V(t_s) \cdot J_{i_s} \cdot D_V(t_{s-1}) \cdot J_{i_{s-1}} \cdot \cdots \cdot D_V(t_1) \cdot J_{i_1} \cdot D_V(t_0).$$

That is, as it is shown in Figure 2, within a time period t, spontaneous decay occurs to each qubit with the same decay rate and among the period quantum jumps occurs s times.

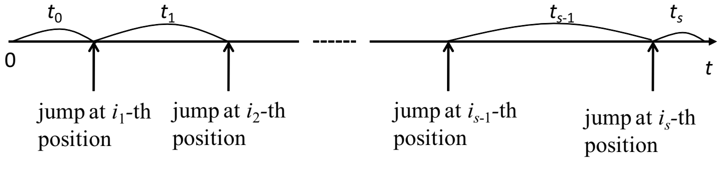

Figure 2. Decay and jump process

2.2. Decoherence-free subspace for decay operator

Our aim in this paper is to construct a code which can correct errors caused by quantum decay and quantum jumps. For quantum decay error, we apply a passive error correction, that is, we consider the error-free space caused by spontaneous decay error operator $D_V(t)$.

Hence, we find a subspace W in which every quantum state vector is invariant with respect to the state transition by $D_V(t)$. A subspace W is called a *decoherence-free subspace* if $D_V(t)|\varphi\rangle = \alpha|\varphi\rangle$ holds for any $|\varphi\rangle \in W$, where α is a nonzero constant.

Now, let $\mathcal{F}_k^n = \{\boldsymbol{x} \in \mathcal{F}^n : \mathrm{wt}(\boldsymbol{x}) = k\}$ and let $W_k =< |\boldsymbol{x}\rangle : \boldsymbol{x} \in \mathcal{F}_k^n >$ be a subspace which is spanned by $\{|\boldsymbol{x}\rangle : \boldsymbol{x} \in \mathcal{F}_k^n\}$.

Lemma 3 *W is a decoherence-free subspace with respect to a decay operator $D_V(t)$ if and only if W is a subspace of W_k for an arbitrary fixed weight k.*

Proof. For any $|\varphi\rangle = \sum_{\boldsymbol{x}\in\mathcal{F}^n} \alpha_{\boldsymbol{x}}|\boldsymbol{x}\rangle \in W$,

$$\begin{aligned} D_V(t)|\varphi\rangle &= \sum_{\boldsymbol{x}\in\mathcal{F}^n} \alpha_{\boldsymbol{x}}^{(i)} D_V(t)|\boldsymbol{x}\rangle \\ &= \sum_{\boldsymbol{x}\in\mathcal{F}^n} \alpha_{\boldsymbol{x}}^{(i)} e^{-\mathrm{wt}(\boldsymbol{x})\frac{\kappa t}{2}}|\boldsymbol{x}\rangle \\ &= \sum_{k=0}^{n} \sum_{\boldsymbol{x}\in\mathcal{F}_k^n} \alpha_{\boldsymbol{x}}^{(i)} e^{-\frac{\kappa k t}{2}}|\boldsymbol{x}\rangle \end{aligned}$$

holds. In order that $D_V(t)|\varphi\rangle = const.|\varphi\rangle$ holds for any t, weight k must be constant, which prove the lemma. □

Hence, any quantum jump code $\mathcal{C}$ must be in a subspace of W_k for some k to ignore the quantum decay error. Furthermore, for a quantum state $|c\rangle \in W_k$ and a jump operator J_E, $J_E|c\rangle \in W_{k'}$ holds for some $k' \leq k$.

2.3. Quantum jump codes

If we want to find an "e-error correcting" quantum jump code as a subspace of W_k, then we have only to consider error operators of the form

$$\mathcal{E} = \{J_E : E \in \mathcal{L}_U, U \subset V, |E| \leq e\}.$$

In quantum jump codes, it is assumed that the positions $E = (i_1, i_2, \ldots, i_s)$ of quantum jump occured are known by the continuous observation of photodetector as it was stated in Section 2.1. Note that if the error positions are known, the conditions (i) and (ii) in Theorem 1 are simplified as

$$\langle c_i|J_E^\dagger J_E|c_j\rangle = \delta_{ij}\kappa_E \quad \text{for any } i \neq j \text{ and } J_E \in \mathcal{E}, \tag{5}$$

where κ_E is a nonzero constant depending only on E.

A subspace of W_k satisfying (5) is called an *e-error correcting quantum jump code*, denoted by an $(n, m, e)_k$ *jump code*, where m is the dimension of $\mathcal{C}$.

3. Examples of 1- and 2-error correcting quantum jump codes

3.1. A 1-error correcting quantum jump code of length four

Here, we consider an example of 1-error correcting quantum jump codes of length four. Let $\mathcal{C}$ be a 1-error correcting quantum jump code. A codeword $|c\rangle$ is represented by

$$|c\rangle = \sum_{\boldsymbol{x} \in \mathcal{F}_k^n} \alpha_{\boldsymbol{x}} |\boldsymbol{x}\rangle$$

for some fixed weight k, $0 \leq k \leq 4$.

Now, let

$$\{c_i = \sum_{\boldsymbol{x} \in \mathcal{F}_k^n} \alpha_{\boldsymbol{x}}^{(i)} |\boldsymbol{x}\rangle : i = 1, 2, \ldots, m\}$$

be an orthonormal basis of $\mathcal{C}$. Since, $\langle c_i | c_j \rangle = \delta_{ij}$,

$$\sum_{\boldsymbol{x} \in \mathcal{F}_k^4} \overline{\alpha}_{\boldsymbol{x}}^{(i)} \alpha_{\boldsymbol{x}}^{(j)} = \delta_{ij} \tag{6}$$

holds for each i and j, and for a fixed $k \in \{0, 1, \ldots, 4\}$, where $\overline{\alpha}_{\boldsymbol{x}}^{(i)}$ is the complex conjugate of $\alpha_{\boldsymbol{x}}^{(i)}$.

Similarly,

$$J_\ell |c_i\rangle = \sum_{\boldsymbol{x} \in \mathcal{F}_k^4} \alpha_{\boldsymbol{x}}^{(i)} J_\ell |\boldsymbol{x}\rangle = \sum_{\boldsymbol{x} \in \mathcal{F}_k^4, x_\ell = 1} \alpha_{\boldsymbol{x}}^{(i)} |P_\ell \boldsymbol{x}\rangle,$$

where P_ℓ is the 4×4 diagonal matrix whose diagonal elements are 1 except for the ℓ-th element being 0. Hence,

$$\langle c_i | J_\ell^\dagger J_\ell | c_j \rangle = \sum_{\boldsymbol{x} \in \mathcal{F}_k^4, x_\ell = 1} \overline{\alpha}_{\boldsymbol{x}}^{(i)} \alpha_{\boldsymbol{x}}^{(j)} = \delta_{ij} \kappa_{k,\ell,1} \tag{7}$$

holds for each i, j and $\ell \in V$, where $\kappa_{k,\ell,1}$ is a nonzero constant depending only on k and ℓ.

Hence, (6) and (7) can be rewrited as

$$\sum_{\boldsymbol{x} \in \mathcal{F}_k^4, x_\ell = 1} \overline{\alpha}_{\boldsymbol{x}}^{(i)} \alpha_{\boldsymbol{x}}^{(j)} = \delta_{ij} \kappa_{k,\ell,1} \tag{8}$$

$$\sum_{\boldsymbol{x} \in \mathcal{F}_k^4, x_\ell = 0} \overline{\alpha}_{\boldsymbol{x}}^{(i)} \alpha_{\boldsymbol{x}}^{(j)} = \delta_{ij} \kappa_{k,\ell,0} \tag{9}$$

for any i, j and ℓ.

Hence, we can easily see that the weight k of the decoherence-free subspace W_k must be 2, since

$$\alpha^{(i)}_{0000} = 0, \quad \alpha^{(i)}_{0001} = \alpha^{(i)}_{0010} = \alpha^{(i)}_{0100} = \alpha^{(i)}_{1000} = 0,$$
$$\alpha^{(i)}_{1111} = 0, \quad \alpha^{(i)}_{1110} = \alpha^{(i)}_{1101} = \alpha^{(i)}_{1011} = \alpha^{(i)}_{0111} = 0$$

hold by (8) and (9).

Moreover we have the following equations for any i and j by (8) and (9):

$$\begin{aligned}
\overline{\alpha}^{(i)}_{1100}\alpha^{(j)}_{1100} + \overline{\alpha}^{(i)}_{1010}\alpha^{(j)}_{1010} + \overline{\alpha}^{(i)}_{1001}\alpha^{(j)}_{1001} &= \delta_{ij}\kappa_{k,1,1},\\
\overline{\alpha}^{(i)}_{1100}\alpha^{(j)}_{1100} + \overline{\alpha}^{(i)}_{0110}\alpha^{(j)}_{0110} + \overline{\alpha}^{(i)}_{0101}\alpha^{(j)}_{0101} &= \delta_{ij}\kappa_{k,2,1},\\
\overline{\alpha}^{(i)}_{1010}\alpha^{(j)}_{1010} + \overline{\alpha}^{(i)}_{0110}\alpha^{(j)}_{0110} + \overline{\alpha}^{(i)}_{0011}\alpha^{(j)}_{0011} &= \delta_{ij}\kappa_{k,3,1},\\
\overline{\alpha}^{(i)}_{1001}\alpha^{(j)}_{1001} + \overline{\alpha}^{(i)}_{0101}\alpha^{(j)}_{0101} + \overline{\alpha}^{(i)}_{0011}\alpha^{(j)}_{0011} &= \delta_{ij}\kappa_{k,4,1},\\
\overline{\alpha}^{(i)}_{0011}\alpha^{(j)}_{0011} + \overline{\alpha}^{(i)}_{0101}\alpha^{(j)}_{0101} + \overline{\alpha}^{(i)}_{0110}\alpha^{(j)}_{0110} &= \delta_{ij}\kappa_{k,1,0},\\
\overline{\alpha}^{(i)}_{0011}\alpha^{(j)}_{0011} + \overline{\alpha}^{(i)}_{1001}\alpha^{(j)}_{1001} + \overline{\alpha}^{(i)}_{1010}\alpha^{(j)}_{1010} &= \delta_{ij}\kappa_{k,2,0},\\
\overline{\alpha}^{(i)}_{0101}\alpha^{(j)}_{0101} + \overline{\alpha}^{(i)}_{1001}\alpha^{(j)}_{1001} + \overline{\alpha}^{(i)}_{1100}\alpha^{(j)}_{1100} &= \delta_{ij}\kappa_{k,3,0},\\
\overline{\alpha}^{(i)}_{0110}\alpha^{(j)}_{0110} + \overline{\alpha}^{(i)}_{1010}\alpha^{(j)}_{1010} + \overline{\alpha}^{(i)}_{1100}\alpha^{(j)}_{1100} &= \delta_{ij}\kappa_{k,4,0}.
\end{aligned}$$

By solving these equations for $i = j$, we can find the following relations:

$$\begin{aligned}
&|\alpha^{(i)}_{1100}| = |\alpha^{(i)}_{0011}| = w^{(i)}_1, \quad |\alpha^{(i)}_{1010}| = |\alpha^{(i)}_{0101}| = w^{(i)}_2, \quad |\alpha^{(i)}_{1001}| = |\alpha^{(i)}_{0110}| = w^{(i)}_3,\\
&{w^{(i)}_1}^2 + {w^{(i)}_2}^2 + {w^{(i)}_3}^2 = const.
\end{aligned} \tag{10}$$

Moreover, in the case of $i \neq j$, we have

$$\begin{aligned}
&\overline{\alpha}^{(i)}_{1100}\alpha^{(j)}_{1100} = \overline{\alpha}^{(i)}_{0011}\alpha^{(j)}_{0011}, \quad \overline{\alpha}^{(i)}_{1010}\alpha^{(j)}_{1010} = \overline{\alpha}^{(i)}_{0101}\alpha^{(j)}_{0101}, \quad \overline{\alpha}^{(i)}_{1001}\alpha^{(j)}_{1001} = \overline{\alpha}^{(i)}_{0110}\alpha^{(j)}_{0110},\\
&\overline{\alpha}^{(i)}_{1100}\alpha^{(j)}_{1100} + \overline{\alpha}^{(i)}_{1010}\alpha^{(j)}_{1010} + \overline{\alpha}^{(i)}_{1001}\alpha^{(j)}_{1001} = 0.
\end{aligned} \tag{11}$$

A solution satisfying (10), (11) can be obtained as follows:

$$|c_i\rangle = u_{i1}|h_1\rangle + u_{i2}|h_2\rangle + u_{i3}|h_3\rangle$$

for $i = 1, 2, 3$, where

$$|h_1\rangle = |1100\rangle + e^{i\theta_1}|0011\rangle, \quad |h_2\rangle = |1010\rangle + e^{i\theta_2}|0101\rangle, \quad |h_3\rangle = |1001\rangle + e^{i\theta_3}|0110\rangle$$

and

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix}$$

is a unitary matrix. In particular, let $\theta_i = 0$ for any i and let $U = I$, then

$$|c_1\rangle = |1100\rangle + |0011\rangle, \quad |c_2\rangle = |1010\rangle + |0101\rangle, \quad |c_1\rangle = |1001\rangle + |0110\rangle$$

is an example of a 1-error correcting quantum jump code of length 4.

It is easy to show that the code $\mathcal{C}$ with an orthonormal basis (ONB) $\{|c_1\rangle, |c_2\rangle, |c_3\rangle\}$ has the maximum dimension. In fact, as we saw that any 1-error correcting jump code $\mathcal{C}$ of length 4 is a subspace of a space spanned by $\{|\boldsymbol{x}\rangle | \boldsymbol{x} \in \mathcal{F}_2^4\}$. Moreover, after a jump error J_ℓ occured, $J_\ell \mathcal{C}$ still has to have the same dimension with $\mathcal{C}$ because $J_\ell|c_i\rangle$ must be orthogonal for any i. Let ℓ be a position where a codeword $|c_1\rangle = \sum_{\boldsymbol{x}\in\mathcal{F}_2^4} \alpha_{\boldsymbol{x}}^{(1)} |\boldsymbol{x}\rangle$ has a term that $\alpha_{\boldsymbol{x}}^{(1)} \neq 0$ and $\boldsymbol{x}$ has 1 in the position ℓ. In this case, $\langle c_1|J_\ell^\dagger J_\ell|c_1\rangle < \langle c_1|c_1\rangle = 1$. Since $\langle c_i|J_\ell^\dagger J_\ell|c_i\rangle = \langle c_1|J_\ell^\dagger J_\ell|c_1\rangle$ holds for any $|c_i\rangle = \sum_{\boldsymbol{x}\in\mathcal{F}_2^4} \alpha_{\boldsymbol{x}}^{(i)} |\boldsymbol{x}\rangle$, there must be a vector $|\boldsymbol{x}\rangle$ such that $|\boldsymbol{x}\rangle$ has 1 at position ℓ and $\alpha_{\boldsymbol{x}}^{(i)} \neq 0$. Thus, $J_\ell \mathcal{C}$ is spanned by the ket vector whose weight is one and $x_\ell = 0$. There are three such vectors of weight 1 whose ℓ-th element is 0. Thus dim $\mathcal{C} \leq 3$ holds.

3.2. *An example of 2-error correcting quantum jump codes of length 6*

Here, we consider an example of 2-error correcting quantum jump codes of length 6.

Let $\mathcal{C}$ be a 2-error correcting quantum jump code. And let

$$\{c_i = \sum_{\boldsymbol{x}\in\mathcal{F}_k^6} \alpha_{\boldsymbol{x}}^{(i)} |x\rangle : i = 1, 2, \dots, m\}$$

be an orthonormal basis of $\mathcal{C}$.

Let $E = \{\ell_1, \ell_2\}$ be the set of positions where jump errors occur. If there are some $\boldsymbol{x}$ such that supp$(\boldsymbol{x}) \supset E$ and $\alpha_{\boldsymbol{x}}^{(i)} \neq 0$, we have

$$\begin{aligned} J_E|c_i\rangle &= \sum_{\boldsymbol{x}\in\mathcal{F}_k^6} \alpha_{\boldsymbol{x}}^{(i)} J_E|\boldsymbol{x}\rangle \\ &= \sum_{\boldsymbol{x}\in\mathcal{F}_k^6, x_\ell = 1 \text{for } \ell\in E} \alpha_{\boldsymbol{x}}^{(i)} |P_E \boldsymbol{x}\rangle, \end{aligned}$$

where P_E is the 6×6 diagonal matrix whose diagonal elements are 1 except for the positions in E being 0. Hence,

$$\langle c_i|J_E^\dagger J_E|c_j\rangle = \sum_{\boldsymbol{x}\in\mathcal{F}_k^6, x_\ell = 1 \text{for } \ell\in E} \overline{\alpha}_{\boldsymbol{x}}^{(i)} \alpha_{\boldsymbol{x}}^{(j)} = \delta_{i,j}\kappa_E, \tag{12}$$

where κ_E is a nonzero constant depending only on E.

By (12), it is shown that $\alpha_{\boldsymbol{x}}^{(i)} = 0$ for any $\boldsymbol{x}$ with wt$(\boldsymbol{x}) = 0, 1, 2, 4, 5, 6$. Thus, in this case a decoherence-free subspace is W_3.

The following is an example of $(6, 2, 2)_3$ jump code.

Example 4 *A* $(6, 2, 2)_3$ *jump code is given by the following orthonormal basis:*

$$|c_1\rangle = \frac{1}{\sqrt{10}}(|111000\rangle + |101100\rangle + |100110\rangle + |100011\rangle + |110001\rangle$$
$$+ |011010\rangle + |001101\rangle + |010110\rangle + |001011\rangle + |010101\rangle),$$
$$|c_2\rangle = \frac{1}{\sqrt{10}}(|000111\rangle + |010011\rangle + |011001\rangle + |011100\rangle + |001110\rangle$$
$$+ |100101\rangle + |110010\rangle + |101001\rangle + |110100\rangle + |101010\rangle).$$

It can be checked that (12) holds for any $E \subset V$, $|E| \leq 2$. For example, let $E = \{1, 2\}$, then

$$J_E|c_1\rangle = \frac{1}{\sqrt{10}}(|001000\rangle + |000001\rangle) \text{ and}$$
$$J_E|c_2\rangle = \frac{1}{\sqrt{10}}(|000010\rangle + |000100\rangle)$$

hold, which imply that $\langle c_i|J_E^\dagger J_E|c_i\rangle = \frac{1}{5}$ for $i = 1, 2$ and $\langle c_1|J_E^\dagger J_E|c_2\rangle = 0$. Similarly, for any E with two elements, $J_E|c_i\rangle$ consists of two basis ket vectors. Also, for any E with a single element, it consists of five basis vectors. These facts implies that

$$\langle c_i|J_E^\dagger J_E|c_j\rangle = \begin{cases} \frac{1}{5}, & \text{if } i = j \text{ and } |E| = 2, \\ \frac{1}{2}, & \text{if } i = j \text{ and } |E| = 1, \\ 1, & \text{if } i = j \text{ and } E = \phi, \\ 0, & \text{if } i \neq j. \end{cases}$$

Remark: As you will see later, $|c_1\rangle$ and $|c_2\rangle$ are derived from two disjoint "2-$(6, 3, 2)$ designs", which include all triples from V.

4. An upper bound for the dimension of jump codes

In this section, fundamental properties of an $(n, m, e)_k$ jump code are described. Most of the results in this section, we refer the reader to Beth *et al.* [8].

For a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{F}^n$ and $E = \{\ell_1, \ell_2, \ldots, \ell_s\}$, let $\overline{\boldsymbol{x}} = (1 - x_1, 1 - x_2, \ldots, 1 - x_n)$ and $\boldsymbol{x}|_E = (x_{\ell_1}, x_{\ell_2}, \ldots, x_{\ell_s})$.

Lemma 5 *Let $\mathcal{C}$ be an $(n, m, t)_k$ jump code. Then for any E, with $|E| = s \leq t$, and for any $\boldsymbol{y} \in \mathcal{F}^s$, we obtain*

$$\sum_{\boldsymbol{x} \in \mathcal{F}_k^n, \boldsymbol{x}|_E = \boldsymbol{y}} \overline{\alpha}_{\boldsymbol{x}}^{(i)} \alpha_{\boldsymbol{x}}^{(j)} = \delta_{ij} \kappa_{E,\boldsymbol{y}}. \tag{13}$$

Proof. In the case of $\boldsymbol{y} = (1, 1, \ldots, 1) \in \mathcal{F}^s$ for $s \leq t$, (13) is ovbious by (5). We prove it by induction for the weight of $\boldsymbol{y}$. For $E \subset V$ such that $|E| = s < t$ and a vector $\boldsymbol{y} \in \mathcal{F}^s$ with weight $w < s$, without loss of generality, we assume that the first w elements are 1 and the other $s - w$ elements are 0. Let $E_0 = \{i : y_i = 1\}$ and we define

$$N(E, \boldsymbol{y}) = \sum_{\boldsymbol{x} \in \mathcal{F}_k^n \boldsymbol{x}_E = \boldsymbol{y}} \overline{\alpha}_{\boldsymbol{x}}^{(i)} \alpha_{\boldsymbol{x}}^{(j)}.$$

Then,

$$N((E, \boldsymbol{y}) = N(E_0, \mathbf{1}_w) - \sum_{\boldsymbol{z} \in \mathcal{F}^{s-w}, \boldsymbol{z} \neq \mathbf{0}} N(E, (\mathbf{1}_w, \boldsymbol{z}))$$

holds. In the case when $i = j$, each term in the right hand side of the above equation is constant by the induction assumption. Similarly, when $i \neq j$, each term in the right hand side is 0, hence the lemma is proved. □

The following lemma is a direct consequence of Lemma 5.

Lemma 6 (Beth *et al.* **[8])** *If C is an $(n, m, t)_k$ jump code, then $\sigma_X^{\otimes n}\mathcal{C}$ is an $(n, m, t)_{(w-k)}$ jump code.*

Lemma 7 (Beth *et al.* **[8])** *If an $(n, m, t)_k$ jump code exists for $k > t > 1$, then an $(n-1, m, t-1)_{(k-1)}$ exists.*

Proof. The lemma can be obtained by applying an error operator J_n to the $(n, m, t)_k$ jump code $\mathcal{C}$. Note that if $\{|c_i\rangle : i = 1, \ldots, m\}$ is an othonormal basis then $\{J_n|c_i\rangle : i = 1, \ldots, m\}$ is also an othonormal basis. □

Lemma 8 (Beth *et al.* **[8])** *If an $(n, m, t)_k$ jump code exists for $k > t \geq 1$, then an $(n+1, m, t)_k$ jump code and an $(n+1, m, t)_{k+1}$ jump code exist.*

Proof. Appending $|0\rangle$ or $|1\rangle$ to a an $(n, m, t)_k$ jump code, an $(n+1, m, t)_k$ or an $(n+1, m, t)_{k+1}$jump code can be obtained, respectively.

The following upperbound is obtained by Beth *et al*. [8].

Proposition 9 (Beth *et al.* **[8])** *The dimension m of a $(n, m, t)_k$ jump code is bounded by*

$$m \leq \min\left\{\binom{n-t}{k-t}, \binom{n-t}{k}\right\} \leq \binom{n-t}{\lfloor n/2 \rfloor - t}. \tag{14}$$

Proof. It is ovbious that $(n, m, 0)_k$ jump code has dimension dim $W_k = \binom{n}{k}$. If $\mathcal{C}$ is an $(n, m, t)_k$ jump code, then by Lemma 7, a $J_E\mathcal{C}$ is an $(n-t, m, 0)_{k-t}$ jump code for $E \subset V$, $|E| = t$. Hence, dim $\mathcal{C} \leq \binom{n-t}{k-t}$. Also, by Lemma 6, an $(n, m, t)_{n-k}$ jump code exists, hence an $(n-t, m, 0)_{n-k-t}$ jump code exists, which means dim $\mathcal{C} \leq \binom{n-t}{n-k-t} = \binom{n-t}{k}$. □

Lemma 10 (Beth *et al.* **[8])** *An $(n, m, 1)_k$ jump code attaing the upperbound (14) exists for any even integer n. In the case, $k = \frac{n}{2}$ and $m = \frac{1}{2}\binom{n}{k}$.*

Proof. Let $|c_{\boldsymbol{x}}\rangle = \frac{1}{\sqrt{2}}(|\boldsymbol{x}\rangle + |\overline{\boldsymbol{x}}\rangle)$ for any $\boldsymbol{x} \in \mathcal{F}^n$, wt$(\boldsymbol{x}) \leq \frac{n}{2}$. Then, The code $\mathcal{C}$ with orthonormal basis $\{|c_{\boldsymbol{x}}\rangle\}$ has dimension $m = \frac{1}{2}\binom{n}{\frac{n}{2}}$. And

$$J_i|c_{\boldsymbol{x}}\rangle = \begin{cases} \frac{1}{\sqrt{2}}|\boldsymbol{x}\rangle, & \text{if } i \in \text{supp}(\boldsymbol{x}) \ , \\ \frac{1}{\sqrt{2}}|\overline{\boldsymbol{x}}\rangle, & \text{if } i \notin \text{supp}(\boldsymbol{x}) \end{cases}$$

holds. Hence $\langle c_{\boldsymbol{x}}|J_i^\dagger J_i|c_{\boldsymbol{x}'}\rangle = \frac{1}{2}\delta_{\boldsymbol{x},\boldsymbol{x}'}$, which prove the lemma.

5. Non-existence of a $(6,3,2)_3$ jump code

By the upperbound (14), we have $\dim\mathcal{C} = m \leq 4$ for an $(6,m,2)_3$ jump code $\mathcal{C}$. Moreover, Beth *et al.* [8] showed that there does not exist a $(6,4,2)_3$ jump code.

Here we show that there does not exist a $(6,3,2)_3$ jump code.

Lemma 11 *There is no* $(6,3,2)_3$ *jump code.*

Proof. Assume that there are three orthonormal vectors $|c_1\rangle, |c_2\rangle, |c_3\rangle$ which span the basis of a $(6,3,2)_3$ jump code. Then, these vectors are linear combinations of twenty vectors in $W = \{|\boldsymbol{x}\rangle : \boldsymbol{x} \in \mathcal{F}^6, \text{wt}(\boldsymbol{x}) = 3\}$. Let

$$|c_i\rangle = \sum_{\boldsymbol{x}\in W} \alpha_{\boldsymbol{x}}^{(i)}|\boldsymbol{x}\rangle$$

for $i = 1,2,3$. Without loss of generality, we choose vectors $|111000\rangle$ and $|000111\rangle$. Then similarly to (8) and (9), we obtain the following equations including the term of $\alpha_{111000}^{(i)}$ and $\alpha_{000111}^{(i)}$:

$$\begin{aligned}
\overline{\alpha}_{111000}^{(i)}\alpha_{111000}^{(j)} + \overline{\alpha}_{110100}^{(i)}\alpha_{110100}^{(j)} + \overline{\alpha}_{110010}^{(i)}\alpha_{110010}^{(j)} + \overline{\alpha}_{110001}^{(i)}\alpha_{110001}^{(j)} &= 0 \\
\overline{\alpha}_{111000}^{(i)}\alpha_{111000}^{(j)} + \overline{\alpha}_{101100}^{(i)}\alpha_{101100}^{(j)} + \overline{\alpha}_{101010}^{(i)}\alpha_{101010}^{(j)} + \overline{\alpha}_{101001}^{(i)}\alpha_{101001}^{(j)} &= 0 \\
\overline{\alpha}_{111000}^{(i)}\alpha_{111000}^{(j)} + \overline{\alpha}_{011100}^{(i)}\alpha_{011100}^{(j)} + \overline{\alpha}_{011010}^{(i)}\alpha_{011010}^{(j)} + \overline{\alpha}_{011001}^{(i)}\alpha_{011001}^{(j)} &= 0 \\
\overline{\alpha}_{000111}^{(i)}\alpha_{000111}^{(j)} + \overline{\alpha}_{001011}^{(i)}\alpha_{001011}^{(j)} + \overline{\alpha}_{001101}^{(i)}\alpha_{001101}^{(j)} + \overline{\alpha}_{001110}^{(i)}\alpha_{001110}^{(j)} &= 0 \\
\overline{\alpha}_{000111}^{(i)}\alpha_{000111}^{(j)} + \overline{\alpha}_{010011}^{(i)}\alpha_{010011}^{(j)} + \overline{\alpha}_{010101}^{(i)}\alpha_{010101}^{(j)} + \overline{\alpha}_{010110}^{(i)}\alpha_{010110}^{(j)} &= 0 \\
\overline{\alpha}_{000111}^{(i)}\alpha_{000111}^{(j)} + \overline{\alpha}_{100011}^{(i)}\alpha_{100011}^{(j)} + \overline{\alpha}_{100101}^{(i)}\alpha_{100101}^{(j)} + \overline{\alpha}_{100110}^{(i)}\alpha_{100110}^{(j)} &= 0.
\end{aligned}$$

Summing up all these equations and by subtracting

$$\sum_{\boldsymbol{x}\in W} \overline{\alpha}_{\boldsymbol{x}}^{(i)}\alpha_{\boldsymbol{x}}^{(j)} = 0,$$

we obtain

$$\overline{\alpha}_{111000}^{(i)}\alpha_{111000}^{(j)} = -\overline{\alpha}_{000111}^{(i)}\alpha_{000111}^{(j)}.$$

for any $i \neq j$. This can be shown for any $\boldsymbol{x} \in W$. Hence,

$$\overline{\alpha}_E^{(i)} \alpha_E^{(j)} = -\overline{\alpha}_{E^c}^{(i)} \alpha_{E^c}^{(j)} \tag{15}$$

for any $E \in \binom{V}{3}$ and $i \neq j$, where $E^c = V \setminus E$ and $\binom{V}{k}$ is the set of k-element subsets of V.

By applying the similar calculation to (8) and (9), we obtain

$$|\alpha_E^{(i)}|^2 + |\alpha_{E^c}^{(i)}|^2 = |\alpha_E^{(j)}|^2 + |\alpha_{E^c}^{(j)}|^2 \tag{16}$$

for any $E \in \binom{V}{3}$ and $i \neq j$.

By multiplying (15) for $(i, j) = (1, 2), (2, 3), (3, 1)$, we have

$$|\alpha_E^{(1)} \alpha_E^{(2)} \alpha_E^{(3)}|^2 = -|\alpha_{E^c}^{(1)} \alpha_{E^c}^{(2)} \alpha_{E^c}^{(3)}|^2,$$

which means

$$\alpha_E^{(1)} \alpha_E^{(2)} \alpha_E^{(3)} = \alpha_{E^c}^{(1)} \alpha_{E^c}^{(2)} \alpha_{E^c}^{(3)} = 0.$$

Case 1. The case of $\alpha_E^{(1)} = 0$ and $\alpha_{E^c}^{(1)} = 0$: In this case, by (16) $\alpha_E^{(i)} = 0$ and $\alpha_{E^c}^{(i)} = 0$ for any $E \in \binom{V}{3}$ and $i = 1, 2, 3$. Hence, $|c_i\rangle = 0$ for any i, contradiction.
Case 2. The case of $\alpha_E^{(1)} = 0$ and $\alpha_{E^c}^{(2)} = 0$: In this case, $\overline{\alpha}_{E^c}^{(1)} \alpha_{E^c}^{(3)} = \overline{\alpha}_E^{(2)} \alpha_E^{(3)} = 0$ holds by (15). The case of $\alpha_{E^c}^{(1)} = 0$ or $\alpha_E^{(2)} = 0$ results in Case 1. Hence, $\alpha_{E^c}^{(3)} = \alpha_E^{(3)} = 0$, which also results in Case 1. Hence, the lemma is proved. □

By Lemma (11), we found that a $(6, 2, 2)_3$ jump code in Example 4 has the maximum possible dimension for $n = 6$, $k = 3$ and $t = 2$.

6. A t-SEED and a jump code

Though the coefficients $\alpha_{\boldsymbol{x}}^{(i)}$ of a ket vector $|c\rangle = \sum_{\boldsymbol{x} \in \mathcal{F}^n} \alpha_{\boldsymbol{x}}^{(i)} |\boldsymbol{x}\rangle$ are complex numbers in general, by restricting the values of $\alpha_{\boldsymbol{x}}^{(i)}$ to 0 and α, where α is a normalizing constant satisfying $\langle c|c\rangle = 1$, the combinatorial structure of quantum jump codes are closely related to combinatorial designs.

Here, we identify a vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{F}^n$ with its support set $B = \mathrm{supp}(\boldsymbol{x}) = \{i : x_i = 1\}$ and $|\boldsymbol{x}\rangle$ with $|B\rangle$. Then a ket vector $|c\rangle = \sum_{\boldsymbol{x} \in \mathcal{F}^n} \alpha_{\boldsymbol{x}} |\boldsymbol{x}\rangle$ is represented by

$$|c\rangle = |\mathcal{B}\rangle = \frac{1}{\sqrt{|\mathcal{B}|}} \sum_{B \in \mathcal{B}} |B\rangle,$$

where $\mathcal{B} = \{\mathrm{supp}(\boldsymbol{x}) : \alpha_{\boldsymbol{x}} \neq 0\}$.

Now, let V/E be the family of subsets of V including $E \subset V$. We define projection matrices

$$M_k = \sum_{\boldsymbol{x} \in \mathcal{F}_k^n} |\boldsymbol{x}\rangle\langle\boldsymbol{x}| = \sum_{B \in \binom{V}{k}} |B\rangle\langle B| \quad \text{and}$$

$$L_E = \sum_{E \subset \text{supp}(\boldsymbol{x})} |\boldsymbol{x}\rangle\langle\boldsymbol{x}| = \sum_{B \in V/E} |B\rangle\langle B|$$

for any $0 \leq k \leq n$ and $E \subset V$, then for any state $|c\rangle = \sum_{\boldsymbol{x} \in \mathcal{F}^n} \alpha_{\boldsymbol{x}} |\boldsymbol{x}\rangle$,

$$M_k|c\rangle = \sum_{\boldsymbol{x} \in \mathcal{F}_k^n} \alpha_{\boldsymbol{x}} |\boldsymbol{x}\rangle = \frac{1}{\sqrt{|\mathcal{B}|}} \sum_{B \in \mathcal{B} \cap \binom{V}{k}} |B\rangle,$$

$$L_E|c\rangle = \sum_{\text{supp}(\boldsymbol{x}) \supset E} \alpha_{\boldsymbol{x}} |\boldsymbol{x}\rangle = \frac{1}{\sqrt{|\mathcal{B}|}} \sum_{B \in \mathcal{B} \cap (V/E)} |B\rangle$$

hold.

By using M_k and L_E, the conditon (5) for a t-error correcting quantum jump code with orthonormal basis $\{|c_i\rangle\}$ can be characterized by

(i) $\langle c_i | M_k | c_i \rangle = 1$ for any i and for given k $(t < k < n - t)$,
(ii) $\langle c_i | L_E | c_j \rangle = \delta_{ij} \lambda_E$ for any i, j and $E \subset V$ such that $|E| \leq t$.

For an orthonormal basis $\{|c_i\rangle = |\mathcal{B}_i\rangle : i = 1, 2, \ldots, m\}$, by noting

$$\langle c_i | L_E | c_j \rangle = \frac{1}{\sqrt{|\mathcal{B}^{(i)}| \cdot |\mathcal{B}^{(j)}|}} \sum_{B \in \mathcal{B}^{(i)} \cap \mathcal{B}(j) \cap (V/E)} 1 = \frac{|\mathcal{B}^{(i)} \cap \mathcal{B}^{(j)} \cap (V/E)|}{\sqrt{|\mathcal{B}^{(i)}| \cdot |\mathcal{B}^{(j)}|}},$$

we find that (i) and (ii) implies

(T1) $|B| = k$ for any $B \in \mathcal{B}^{(i)}$,
(T2) $\frac{|\{B \in \mathcal{B}^{(i)} : B \supset E\}|}{|\mathcal{B}^{(i)}|} = \lambda_E$ holds for any i and $E \subset V$, $|E| \leq t$, where λ_E is a constant depending on E.
(T3) $\mathcal{B}^{(i)} \cap \mathcal{B}^{(j)} = \phi$ for $i \neq j$.

For an n-set V and $\mathcal{B}^{(i)} \subset \binom{V}{k}$, $(i = 1, \ldots, m)$, if (T1), (T2), (T3) are satisfied, then a system $(V; \mathcal{B}^{(1)}, \ldots \mathcal{B}^{(m)})$ is called a *t-spontaneous emission error design*, denoted by t-$(n, k; m)$ SEED (see Figure 3).

Note that when λ_E depends only on the number of elements in $|E|$, a pair $(V, \mathcal{B}^{(i)})$ is called a t-(n, k, λ) *design*, where $\lambda = \lambda_E$ for $|E| = t$. In particular, a t-$(n, k, 1)$ design is called a *Steiner t-design*, denoted by $S(t, k, v)$. Moreover if $|\mathcal{B}|$ is constant and $\bigcup_{i=1}^m \mathcal{B}^{(i)} = \binom{V}{k}$, a t-SEED is called a *large set* of a t-(n, k, λ) design, denoted by $\text{LS}_\lambda(t, k, n)$. The number of t-designs in a large set is $m = \binom{v-t}{k-t} / \lambda$.

Lemma 12 *For a fixed $k \leq \frac{n}{2}$, an $\text{LS}_1(t, k, n)$ attains the upper bound (14) of Proposition 9.*

7. Constructions of t-SEEDs

In this section various constructions of t-SEEDs are described.

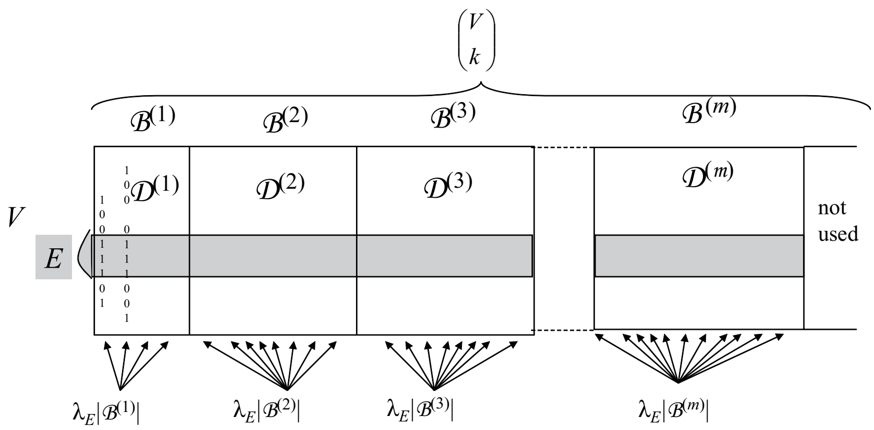

Figure 3. An incidence matrix of a t-SEED

7.1. Large sets

Firstly, known large sets are listed here. For deatils of large sets, we refer the reader to Khosrovshahi and Tayfeh-Rezaie [20], Colbourn and Dinitz [13] and Tierlinck [31].

(i) An $\text{LS}_1(2, 3, v)$ exists for all admissible parameters of $v \neq 7$.
(ii) An $\text{LS}_{\lambda_{\min}}(3, 4, v)$ exists for $v \equiv 0 \pmod 3$.
(iii) An $\text{LS}_{\lambda_{\min}}(4, 5, 20v + 4)$ exists for $\gcd(v, 30) = 1$.
(iv) An $\text{LS}_{60}(4, 5, 60v + 4)$ exists for $\gcd(v, 60) = 1, 2$.
(v) The number of disjoint designs in $\text{LS}_\lambda(t, t + 1, v)$ is $\ell = \frac{v-t}{\lambda}$.
(vi) No $\text{LS}_1(3, 4, v)$ is known.
(vii) Etizon and Hartman(1991) obtained near large set with $v - 5$ disjoint 3-$(v, 4, 1)$ designs for $v = 5 \cdot 2^n$.
(viii) An $\text{LS}_3(3, 4, v)$ exists for $v \equiv 0, 6 \pmod{12}$.
(ix) An $\text{LS}_6(3, 4, v)$ exists for $v \equiv 9 \pmod{12}$.
(x) An $\text{LS}_{12}(3, 4, v)$ exists for $v \equiv 3 \pmod{12}$.

7.2. t-SEEDs derived from orthogonal arrays

Let S be a set of q elements. A $q^t \times k$ array A with elements in S is called an *orthogonal array*, denoted by $\text{OA}(t, k, q)$, if each ordered t-tuple occurs exactly once in any t-columns of A. A large set of an orthogonal array $\text{LOA}(t, k, q)$ is a collection $\{A_r\}_{r \in R}$ of $\text{OA}(t, k, q)$'s such that every ordered k-tuple of S occurs exactly once in one of A_r. Note that $|R| = q^{k-t}$.

The following is known (see Raghavarao [26]):

Proposition 13 *If there is an OA(t, k, q), then there is a large set LOA(t, k, q).*

By this Proposition, we obtain the following:

Theorem 14 *If there exists an OA(t, k, q), then there exists a t-$(kq, k; q^{k-t})$ SEED.*

Example 15 *If q is a prime power, then there exists a t-$(qk, k; q^{k-t})$ SEED for $k \leq q+1$.*

Remark: Beth *et al*. [8] obtained a t-SEED for $k = q$. Moreover, Beth et al. [8] claimed that

$$\frac{\log(\text{dim. of jump code by Theorem 1})}{\log(\text{the upper bound of (14)})} = \frac{(q-t)\log q}{\log \binom{q^2-t}{q-t}} \longrightarrow 1$$

as $q \longrightarrow \infty$ for fixed t. On the other hand, it holds that

$$\frac{\text{dim. of jump code by Theorem 1}}{\text{the upper bound of (14)}} = \frac{q^{q-t}}{\binom{q^2-t}{q-t}} \longrightarrow 0 \tag{17}$$

as $q \longrightarrow \infty$ for fixed t.

Hence, we may pose a question whether there is a sequence of t-SEEDs which is asymptotically optimal in the sence that (17) tend to 1 except for a series of large sets?

7.3. Product methods and recursive constructions

Let $(V_1; \mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(m)})$ be a t-$(n, k; m)$ SEED and $k \times q^t$ matrices $A^{(\ell)} = (a_{ij}^{(\ell)})$ be an LOA(t, k, q) with elements $\{0, 1, \ldots, q-1\}$. Let $V = V_1 \times \{0, 1, \ldots, q-1\}$ and construct families of blocks

$$\mathcal{B}^{(h,\ell)} = \{\{(b_1, a_{1j}^{(\ell)}), \ldots, (b_k, a_{kj}^{(\ell)})\} : (b_1, \ldots, b_k) \in \mathcal{B}^{(h)},\ j = 1, \ldots, q^t\}$$

for $h = 1, \ldots, m, \ell = 1, \ldots, q^{k-t}$. Then, we obtain the following theorem.

Theorem 16 *If there are a t-$(n, k; m)$ SEED and a LOA(t, k, q), then there is a t-$(nq, k; mq^{k-t})$ SEED.*

Applying this recusive construction to Theorem 14, we obtain the following:

Corollary 17 *For a prime power q, a t-$(q^n(q+1), q+1; q^{n(q+1-t)})$ SEED exists.*

Beth *et al.* [8] gave a construction which combines a quantum jump code and a usual quantum code.

Theorem 18 (Beth et al. [8]) *Let $\mathcal{C} = (n, p, t)_k$ be a jump code of prime dimension. Furthermore, let $\mathcal{C}_p = [[N, K, D]]_p$ be a "quantum error-correcting code" in the space $(\mathbb{C}^p)^{\otimes N}$. Then the concatenation of $\mathcal{C}$ as inner and $\mathcal{C}_p$ as outer code yields a jump code $\overline{\mathcal{C}} = (Nn, pK, T)_{Nw}$ on Nn-qubits with $T \geq D(t+1) - 1$.*

A t-$(n, k; m)$ SEED $(V; \mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(m)})$ is said to be *s-resolvable* if each $\mathcal{B}^{(i)}$ is partitioned into h subfamilies $\mathcal{B}^{(i,1)}, \ldots, \mathcal{B}^{(i,h)}$ and a $(V; \mathcal{B}^{(1,1)}, \mathcal{B}^{(1,2)}, \ldots, \mathcal{B}^{(m,h)})$ forms an s-$(n, k; mh)$ SEED.

Theorem 19 *If there is a $\lfloor \frac{t}{2} \rfloor$-resolvable t-$(n, k; m)$ SEED $(V; \mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(m)})$, then there exists a t-$(nv, 2k; hm^2)$ SEED for any $v \geq 2$, where h is the number of subfamilies $\mathcal{B}^{(i,j)}$ in $\mathcal{B}^{(i)}$.*

We will give an example of Theorem 19. Let K_n be the complete graph of order n. For even n, a 1-factor of K_n is a set of independent edges. A 1-factorization of K_n is a partition of the edges of K_n into $n-1$ one-factors. For any even n, there exists a 1-factorization of K_n.

A 1-factorization can be seen as a 1-resolvable 3-$(n,2;1)$ SEED. Hence, by Theorem 19, we obtain the following corollary.

Corollary 20 *For any even n and for any integer $v \geq 2$, there is a 3-$(nv,4;n-1)$ SEED, which is an $(nv,n-1,3)_4$ jump code.*

Remark: In this case, the upper bound of the dimension is $nv-3$. When $v \leq 3$ this is better than that of Corollary 2 for $k=4$, $t=3$.

Example 21 *In Figure 4 and Table 1, a 1-factor for K_4 is presented. A column of Table 1 corresponds to an edge. And any two columns partitioned by vertical lines correspond to 1-factors. In Table 2, each column corresponds to a block. Let $V = \{0_0,1_0,2_0,3_0,0_1,1_1,2_1,3_1\}$, then a four tuple $(a,b : c,d)$ means a block* $\{a_0,b_0,c_1,d_1\}$.

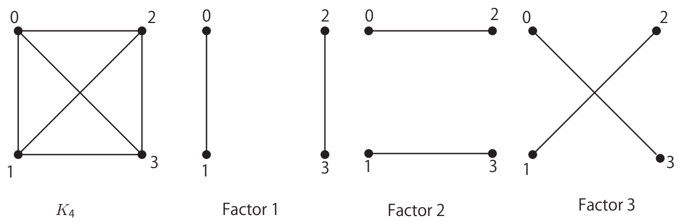

Figure 4. A 1-factor for $n=4$

Table 1. A 1-factor for $n=4$

0	2	0	1	0	1
1	3	2	3	3	2

Table 2. A 3-$(8,4;3)$ SEED for $n=4$, $v=2$, $m=1$, $h=n-1=3$

0 0 2 2 0 0 1 1 0 0 1 1	0 0 2 2 0 0 1 1 0 0 1 1	0 0 2 2 0 0 1 1 0 0 1 1
1 1 3 3 2 2 3 3 3 3 2 2	1 1 3 3 2 2 3 3 3 3 2 2	1 1 3 3 2 2 3 3 3 3 2 2
0 2 0 2 0 1 0 1 0 1 0 1	0 1 0 1 0 1 0 1 0 2 0 2	0 1 0 1 0 2 0 2 0 1 0 1
1 3 1 3 2 3 2 3 3 2 3 2	2 3 2 3 3 2 3 2 1 3 1 3	3 2 3 2 1 3 1 3 2 3 2 3

7.4. 2- and 3-SEEDs derived from affine geometry

It is well known that the set of planes in $\mathrm{AG}(n,q)$ yields a 2-$(v = q^n, k = q^2, \lambda = (q^{n-1}-1)/(q-1))$ design.

Lemma 22 *The* 2*-design generated by the set of 2-flats in AG*(n,q) *is decomposed into*

(i) $\frac{q^{n-1}-1}{q^2-1}$ *number of* 2-$(v = q^n, k = q^2, \lambda = q+1)$ *designs when* n *is odd.*
(ii) $\frac{q^{n-1}-q}{q^2-1}$ *number of* 2-$(v = q^n, k = q^2, \lambda = q+1)$ *designs and one* 2-$(v = q^n, k = q^2, \lambda = 1)$ *design when* n *is even.*

Munemasa [25] showed better results for $q = 2$ by examining the orbit structure of $\mathrm{PG}(2n-1,2)$.

Lemma 23 (Munemasa [25]) *The number of lines in PG*$(2n-1,2)$ *whose orbit under the subgroup of index* 3 *in the Singer group is a spead is given by*

$$\frac{1}{27}(2^{2n}-1)(2^n+(-1)^{n+1})^2. \tag{18}$$

By using Lemma 23, we can obtain a 2-SEED.

Example 24 *PG*$(7,2)$ *has* $2^8-1 = 255$ *points and*

$$\frac{(2^8-1)(2^7-1)}{(2^2-1)(2-1)} = 255 \times 42 + 85$$

lines. These lines are partitioned into 42 Singer cycles whose orbits are full and a single cycle of short orbit with orbit length 85. For a line L *in a full orbit,*

$$\{a(L \cup \mathbf{0}) + b : a \in GF(2^8)^\times, b \in GF(2^8)\}$$

generates a 2-$(2^8,4,3)$ *design and a line in the short orbit generates a* 2-$(2^8,4,1)$ *design. Among those* 42 *full orbits there are* 8 *orbits each of which can be partitioned into 3 spreads of lines. Actually, for a root* β *of the primitive irreducible polynomial* $x^8+x^5+x^3+x^2+x+1$*,* $B = \{\beta^0, \beta^7, \beta^{173}\}$ *and its Frobenius cycle of length* 8 *are lines in such orbits. Hence, we obtain* 24 *spreads and each of these spreads generates a* 2-$(2^8,4,1)$ *design. As a total, we obtain* $(24+1)$ 2-$(2^8,4,1)$ *designs and* $(42-8)$ 2-$(2^8,4,3)$ *designs, which generate a* 2-$(2^8,4;59)$ *SEED.*

In general, by Lemmas 22 and 23 it holds that the number of full orbits are $\frac{2^{2n-1}-2}{3}$. Among these, there are

$$\frac{(2^n+(-1)^{n+1})^2-9}{27}$$

orbits which can be partitioned into 3 spreads. Hence, we obtain the following theorem.

Theorem 25 *The* 2*-design generated by the set of 2-flats in* $AG(2n, 2)$ *is decomposed into*

$$\frac{2^{2n-1}-2}{3} - \frac{(2^n + (-1)^{n+1})^2 - 9}{27}$$

number of disjoint 2-$(2^{2n}, 4, 3)$ *designs and*

$$\frac{(2^n + (-1)^{n+1})^2}{9} - 1$$

number of disjoint 2-$(2^{2n}, 4, 1)$ *designs. Hence, there is a* 2-$(2^{2n}, 4; f_{2n})$ *SEED, where*

$$f_{2n} = \frac{2^{2n-1}-2}{3} + 1 + \frac{2\{(2^n + (-1)^{n+1})^2 - 9\}}{27}.$$

Now, for $V = \text{GF}(2)^n$, let σ be a mapping such taht $\sigma : x \longmapsto x^s$ for $x \in V$. Then our problem is to find the condition on s in order that a $\mathcal{D}$ and $\sigma(\mathcal{D})$ are disjoint, where $\mathcal{D}$ is a 3-design generated from 2-flats of AG$(n, 2)$. When $s = 2^i$, $\sigma : x \longmapsto x^{2^i}$ is a Frobenius automorphism of $\mathcal{D}$. In this case, it holds that $\sigma(\mathcal{D}) = \mathcal{D}$.

Let $\mathcal{D} = (V, \mathcal{B})$ is the 3-$(2^f, 4, 1)$ design derived from 2-flats of AG$(f, 2)$. Let s be an integer such that $(s, n) = 1$, where $n = 2^f - 1$. Then $\sigma : x \longmapsto x^s$ is a bijection on $V = \text{GF}(2^f)$ and $\sigma(\mathcal{D}) = \mathcal{D}^s$ is isomorphic to $\mathcal{D}$.

Lemma 26 *If* $s \in \mathbb{Z}_n$ *satisfies*

(i) $gcd(s, n) = 1$,
(ii) $\forall x \in GF(2^n) \setminus \{0, 1\}, (1 + x)^s \neq 1 + x^s$,
(iii) $\forall x \neq \forall y \in GF(2^n) \setminus \{0, 1\}, (1 + x + y)^s \neq 1 + x^s + y^s$,

then the designs $\mathcal{D}^s$ *and* $\mathcal{D}$ *are isomorphic and disjoint.*

Lemma 27 *When* f *is odd,* $s = 3$ *satisfies the conditions (i), (ii), (iii) of Lemma 26*

Remark:If s satisfies the condition (i), (ii), (iii), then $2^i s$ and $s^{-1} \pmod{n}$ also does. When $m \leq 12$ is even, $s = 1$ is the only parameter satisfying the condition (i) and (ii).

Assume that s and s' satisfy the condition of Lemma 26. Then $\mathcal{D}^s$ and $\mathcal{D}^{s'}$ are also disjoint when $s's^{-1} \pmod{n}$ satisfies the conditon. By choosing a set S such that $s's^{-1} \pmod{n}$ satisfies the condition for each $s, s' \in S$, we obtain a set of disjoint $\mathcal{D}^s$'s.

Example 28 (i) *For* $f = 5$, $s = 1, 3, 5, 7, 11, 15$ *generate six disjoint* 3-$(2^5, 4, 1)$ *designs.*
(ii) *For* $f = 7$, $s = 1, 3, 5, 9, 15, 43$ *generate six disjoint* 3-$(2^7, 4, 1)$ *designs.*

Hence, we obtain the following t-SEEDs:

Lemma 29 *There exists a* 3-$(2^5, 4; d_f)$ *SEED containing a* 2-$(2^5, 4; d_f \frac{2^{f-1}-1}{3})$ *SEED, where* $d_f = 2, 6, 6, \ldots$ *for* $f = 3, 5, 7, \cdots$.

Table 3. List of s such that $\mathcal{D}$ and $\mathcal{D}^s$ are disjoint

m	# of s	representatives of s
3	2	1, 3
5	6	1, 3, 5, 7, 11, 15
7	12	1, 3, 5, 9, 11, 13, 15, 23, 27, 29, 43, 63
9	14	1, 3, 5, 13, 17, 19, 27, 31, 47, 59, 87, 103, 171, 255
11	24	1, 3, 5, 9, 13, 17, 33, 35, 43, 57, 63, 95, 107, 117, 143, 151, 231, 249, 315, 365, 411, 413, 683, 1023
13	28	1, 3, 5, 9, 13, 17, 33, 57, 65, 67, 71, 127, 171, 191, 241, 287, 347, 367, 635, 723, 911, 1243, 1245, 1453, 1639, 1691, 2731, 4095

7.5. 5-SEEDs derived from Golay code

In this section, we review mutually disjoint 5-designs related to the Golay code and self dual codes.

Kramer and Magliveras [21] constructed 9 mutually disjoint Steiner systems $S(5, 8, 24)$ by finding 8 permutations on 24 points. Araya [4] also constructed 15 mutually disjoint Steiner systems $S(5, 8, 24)$ by a computer search. The following results were shown by Jimbo and Shiromoto [18].

Theorem 30 *There exists at least* 22 *mutually disjoint Steiner systems* $S(5, 8, 24)$*. Hence a* 5-$(24, 8; 22)$ *SEED exists.*

Theorem 30 can be obtained by making disjoint isomorphic 22 5-$(24, 8, 1)$ designs from the Golay code. We will give a breif proof of Theorem 30.

Let $\mathcal{G}_{24}$ be the binary extended Golay $[24, 12, 8]$ code with parity-check matrix

$$H(\mathcal{G}_{24}) = \left(\begin{array}{cccccccccccccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&1&0&0&0&1&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&1&0&0&0&1&1\\
0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1&1&1&0&0&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1&1&1&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1&1&1&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&1&0&1&1&1&1\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&1&0&1&1&0&1&1&1\\
0&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0&1&0&1&1&0&1&1\\
0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&1&1&0&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&1&0&0&0&1&0&1&1&1\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&1&1&0&0&0&1&0&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&0
\end{array}\right) = \left(\begin{array}{ccc} & & 1\\ I_{12} & A_{11} & \vdots\\ & & 1\\ & 1\ \cdots\ 1 & 0 \end{array}\right).$$

Note that A_{11} is a circulant matrix, and the Hamming distance between any two distinct row vectors of A_{11} is 6.

Let $\sigma = (13, 14, \ldots, 23)$ and $\tau = (1, 13)(2, 14) \cdots (11, 23)$ be the coordinate permutations which act on the vector space $\mathrm{GF}(2)^{24}$. We denote the zero vector and the all-one vector by **0** and **1**, respectively. For any positive integer m, let J_m be the all-one $m \times m$ matrix. The following lemma is well-known and is essential (see, for instance, Ch. 16 in [22]).

Lemma 31 *Let* X *be a circulant matrix of first row* $(c_0, c_1, \ldots, c_{n-1})$ *over a finite field.* X *is invertible if and only if* $a_0(x) = c_0 + c_1x + \cdots + c_{n-1}x^{n-1}$ *is relatively prime to* $x^n - 1$.

Now the following three lemmas are obtained.

Lemma 32 *For any* $i, j \in \{0, 1, \dots, 10\}$, $i \neq j$, *the intersection between all the codewords in* $\mathcal{G}_{24}^{\sigma^i}$ *and* $\mathcal{G}_{24}^{\sigma^j}$ *is* $\{\mathbf{0}, \mathbf{1}, \boldsymbol{x}, \boldsymbol{x} + \mathbf{1}\}$, *where* $\boldsymbol{x}$ *is the weight* 12 *vector* $(0, \dots, 0, 1, \dots, 1, 0)$.

Lemma 33 *For any* $i, j \in \{0, 1, \dots, 10\}$, $i \neq j$, *the intersection between all the codewords in* $\mathcal{G}_{24}^{\tau\sigma^i}$ *and* $\mathcal{G}_{24}^{\tau\sigma^j}$ *is* $\{\mathbf{0}, \mathbf{1}, \boldsymbol{y}, \boldsymbol{y} + \mathbf{1}\}$, *where* $\boldsymbol{y}$ *is the weight* 12 *vector* $(1, \dots, 1, 0, \dots, 0)$.

Lemma 34 *For any* i *and* j, *the intersection between all the codewords in* $\mathcal{G}_{24}^{\sigma^i}$ *and* $\mathcal{G}_{24}^{\tau\sigma^j}$ *is* $\{\mathbf{0}, \mathbf{1}\}$.

By summarizing these results, we have the following:

Theorem 35 *Let* $\sigma = (13, 14, \dots, 23)$ *and* $\tau = (1, 13)(2, 14) \cdots (11, 23)$ *be the permutations on* 24 *points and let* H *be the set of all permutations of the form* $\tau^l \sigma^i$ *in the permutation group* S_{24}. *And let* $\mathcal{B}$ *be the set of supports of all the Hamming weight* 8 *codewords in* $\mathcal{G}_{24}$. *Then* $\{\mathcal{B}^g \ : \ g \in H\}$ *forms the set of* 22 *mutually disjoint Steiner systems* $S(5, 8, 24)$.

In Theorem 35, for any subset K of H, the collection $\bigcup_{g \in K} \mathcal{B}^g$ can be viewed as a set of blocks in a simple 5-$(24, 8, |K|)$ design. Then we have the following result as a corollary of Theorem 35.

Corollary 36 *There exist simple* 5-$(24, 8, m)$ *designs, for* $m = 1, 2, \dots, 22$.

It is also known that the set of supports of the codewords of Hamming weight 12 in $\mathcal{G}_{24}$ forms a 5-$(24, 12, 48)$ design. From Proposition 34, there is no codewords of Hamming weight 12 in the intersection between $\mathcal{G}_{24}^{\sigma^i}$ and $\mathcal{G}_{24}^{\tau\sigma^j}$.

Corollary 37 *There exists at least two mutually disjoint* 5-$(24, 12, 48)$ *designs. And there exist simple* 5-$(24, 12, 48m)$ *designs, for* $m = 1, 2$.

Recently, Araya and Harada [5] found the following by a computer search.

Theorem 38 (Araya and Harada [5]) *There exists at least* 50 *mutually disjoint Steiner systems* $S(5, 8, 24)$. *Hence a* 5-$(24, 8; 50)$ *SEED exists.*

Theorem 39 (Araya and Harada [5]) *There exists at least* 35 *mutually disjoint 5-*$S(24, 12, 48)$ *designs. Hence a* 5-$(24, 12; 35)$ *SEED exists.*

Similar resluts were obtained for a quadratic residue code of length 48.

Theorem 40 (Jimbo and Shiromoto [18]) *There exists at least* 46 *mutually disjoint simple* 5-$(48, 12, 8)$ *designs. Hence a* 5-$(48, 8; 46)$ *SEED exists.*

The above results are based on a binary extended Golay code of length 24 and a quadratic residue code of length 48. Angata and Shiromoto [3] and Araya, Harada, Tonchev and Wassermann [6] independently generalized the results to the case of Pless symmetry (ternary) code.

Theorem 41 (Angata and Shiromoto [3]) *There exist at least*

(i) 34 *mutually disjoint* 5-$(36,k,\lambda)$ *designs for each* $(k,\lambda)=(12,45)$, $(15,5577)$.
(ii) 58 *mutually disjoint* 5-$(60,k,\lambda)$ *designs, for each* $(k,\lambda) = (18,3060)$, $(21,449820)$, $(24,34337160)$, $(27,1271766600)$.

Remark: Araya, Harada, Tonchev and Wassermann [6] found 17 mutually disjoint 5-$(36,12,45)$ designs.

Theorem 42 (Angata and Shiromoto [3], Araya, Harada, Tonchev, Wassermann [6]) *There exist at least 11 mutually disjoint* 5-$(24,9,6)$ *designs.*

Theorem 43 (Angata and Shiromoto [3]) *There exist at least 23 mutually disjoint* 5-$(48,k,\lambda)$ *designs, for each* $(k,\lambda)=(15,364)$, $(18,50456)$, $(21,2957388)$.

By these results, the following is obtained:

Corollary 44 *There exist*

(i) *a* 5-$(36,12;34)$ *SEED for* $k=12,15$,
(ii) *a* 5-$(60,k;58)$ *SEED for* $k=18,21,24,27$,
(iii) *a* 5-$(24,9;11)$ *SEED, and*
(iv) *a* 5-$(48,k;23)$ *SEED for* $k=15,18,21$.

Moreover, Araya, Harada, Tonchev and Wassermann [6] obtained the following.

Theorem 45 (Araya, Harada, Tonchev and Wassermann [6]) *There exist at least*

(i) *3 mutually disjoint* 5-$(18,8,6)$ *designs,*
(ii) *5 mutually disjoint* 5-$(24,10,36)$ *designs,*
(iii) *2 mutually disjoint* 5-$(25,9,30)$ *designs,*
(iv) *2 mutually disjoint* 5-$(30,12,220)$ *designs,*
(v) *4 mutually disjoint* 5-$(32,6,3)$ *designs.*
(vi) *4 mutually disjoint* 5-$(33,7,4)$ *designs.*

Corollary 46 *There exist a* 5-$(18,8;2)$ *SEED, a* 5-$(24,10;5)$ *SEED, a* 5-$(25,9;2)$ *SEED, a* 5-$(30,12;2)$ *SEED, a* 5-$(32,6;4)$ *SEED and a* 5-$(33,7;4)$ *SEED.*

7.6. More SEEDs from codes

By Assmus and Matson [7]'s theorem, codewords of weight k of codes in Table 4 form 3-designs, or 5-designs. If we can partition the design into subdesigns, 3-SEEDs can be obtained. The results in Table 4 were reported by Shiromoto [27].

From these computation results, the following theorem is shown.

Theorem 47 *There exist a* 3-$(32,8;3)$ *SEED, a* 3-$(32,10;24)$ *SEED, a* 3-$(32,12;52)$ *SEED, a* 3-$(32,14;90)$ *SEED, and a* 3-$(32,16;132)$ *SEED.*

Table 4. Partition of t-designs derived from codes

codes	$\mathrm{Aut}(C)$	weights	designs[*1]	λ's of subdesigns
Extended BCH $[32, 21, 6]$ Code	$\mathrm{A\Gamma L}(1, 32)$	6	3-(32,6,4)	4
		8	3-(32,8,119)	56, 56, 7
		10	3-(32,10,1464)	120×24
		12	3-(32,12,10120)	220×43, 44, 22×3, 110×5
		14	3-(32,14,32760)	364×90
		16	3-(32,16,68187)	560×119, 112×5, 140×7, 7
It's dual $[32, 11, 12]$ Code	$\mathrm{A\Gamma L}(1, 32)$	12	3-(32,12,22)	22
		16	3-(32,16,119)	7,112
Extended BCH $[32, 16, 8]$ Code	$\mathrm{AGL}(2, 5)$	12	3-(32,12,616)	616
		16	3-(32,16,4123)	3136, 7, 980
Self-Dual Extended QR $[32, 16, 8]$ Code	$\mathrm{PSL}(2, 31)$	8	3-(32,8,7)	7
		12	3-(32,12,616)	11, 165, 110, 330
		16	3-(32,16,4123)	112, 336, 560, 840, 210, 140, 105 ,840 ,560 ,420
Self-Dual Extended QR $[48, 24, 12]$ Code	$\mathrm{PSL}(2, 47)$	12	5-(48,12,8)	3-(48,12,λ) 110,55,55
		16	5-(48,16,1365)	unknown
		20	5-(48,20,36176)	unknown

(*1) Assmus & Matson (1969)

(*2) Computations of subdesigns using MAGMA were assisted by M. Angata

8. Concluding remark and open problems

In this paper, we considered constructions of t-error correcting jump codes and t-SEEDs. Besides the construction of t-SEEDs reviewed in this paper, more constructions are presented in Beth *et al.* [8] and Charnes and Beth [12]. Beth *et al.* [8] gave a construction of $(n, 2, t)_k$ jump codes by using isodual binary codes, which was extended by Charnes and Beth [12] utilizing a group theoretical technique. However, only a few results are known for optimal t-SEEDs attaining the upperbound (16) for $t \geq 2$. In general, t-SEEDs have weaker combinatorial conditions than that of large sets. But we do not know any example of optimal t-SEEDs except for large sets.

Problem 48 *Is there an optimal t-SEED attaining the upperbound (16) for $t \geq 2$ except for large sets?*

A jump code can be considered as a continuous version of a t-SEED or a system of disjoint t-designs. Actually, "balancedness" is generalized to the constancy of inner product. Whereas, "disjointness" corresponds to orthogonality.

It is ovbious that if there is a t-$(n, k; m)$ SEED, then there is a $(n, m, t)_k$ jump code. But it may not be known whether there is an example such that there is an $(n, m, t)_k$ jump code even if there is no t-$(n, k; m)$ SEED.

Problem 49 *Is there an* $(n, m, t)_k$ *jump code even if there is no* t-$(n, k; m)$ *SEED. In paticular, a* $(7, 3, 2)_3$ *jump code can be constructed by two disjoint* 2-$(7, 3, 1)$ *designs and one* 2-$(7, 3, 3)$ *designs. But the upperbound for* m *is 5. Is there a* $(7, m, 2)_3$ *jump code for* $m = 4$ *or* 5*?*

Problem 50 *If there is an LS*$_1(t, k, n)$ *it is optimal in the sense that it attains the upperbound (14). However, in the case when there is no LS*$_1(t, k, n)$*, can we find an optimal or asymtotically optimal* t-$(n, k; m)$ *SEED for* $t \geq 2$*?*

In Subsection 7.5, we showed that there are 22 disjoint 5-$(24, 8, 1)$ designs. Whereas, Harada [16] found 50 disjoint 5-$(24, 8, 1)$ designs by computer search. If an $\mathrm{LS}_1(5, 8, 24)$ exists it must have $3 \times 17 \times 19$ disjoint 5-$(24, 8, 1)$ designs. Similarly, we wonder whether a $\mathrm{LS}_{48}(5, 12, 24)$ exists, or not. We pose here a challenging problems.

Problem 51 *Does there exist an LS*$_1(5, 8, 24)$*?*

Problem 52 *Does there exist an LS*$_{48}(5, 12, 24)$*?*

Acknowledgements

The first author wish to thank the ASI and the organizers, Professor Dean Crnković and Professor Vladimir Tonchev, for giving me a chance to attend such an extreamly well-organized conference and for their support.

References

[1] G. Alber, T. Beth, C. Charnes, A. Delgado, M. Grassl and M. Mussinger, Stabilizing distinguishable qubits against spontaneous decay by detected-jump correcting quantum codes, *Physical Review Letters*, **86** (2001) 4402–4405.

[2] G. Alber, T. Beth, C. Charnes, A. Delgado, M. Grassl and M. Mussinger, Detected-jump-error-correcting quantum codes, quantum error designs, and quantum computation. *Physical Review A*, **68** (2003), 012316.

[3] M. Angata and K. Shiromoto, Mutually disjoint 5-designs from Pless symmetry codes, submitted to *J. Statist. Theory and Practice* (2010).

[4] M. Araya, More mutually disjoint Steiner systems $S(5, 8, 24)$, *J. Combin. Theory, Ser. A* **102** (2003), 201–203.

[5] M. Araya and M. Harada, Mutually disjoint Steiner systems $S(5, 8, 24)$ and 5-$(24, 12, 48)$ designs, *Electronic J. Combinatorics*, **17** (2010) N1.

[6] M. Araya, M. Harada, V. D. Tonchev, A. Wassermann, Mutually disjoint designs and new 5-designs derived from groups and codes, to appear in *J. Combin. Designs*, (2010).

[7] E. F. Assmus Jr. and H. F. Mattson Jr., (1969), New 5 designs, J. Combin. Theory **6**, 122–151.

[8] T. Beth, C. Charnes, M. Grassl, G. Alber, A. Delgado, M. Mussinger, A new class of designs which protect against quantum jumps. *Designs, Codes and Cryptography*, **29** (2003), 51–70.

[9] A. R. Calderbank and P. W. Shor, Good quantum error-correcting codes exist, *Physical Review A*, **54**, (2) (1996) 1098–1105.

[10] A.R. Calderbank, E.M. Rains, P.W. Shor, N.J.A. Sloane, Quantum error correction and orthogonal geometry, *Phys. Rev. Lett.*, **78** (1997) 405–408.

[11] A.R. Calderbank, E.M. Rains, P.W. Shor, N.J.A. Sloane, Quantum error correction via codes over GF(4), *IEEE Trans. Inform. Theory* **44** (4) (1998) 1369–1387.

[12] C. Charnes and T. Beth, Combinatorial aspects of jump codes, *Discrete Math.*, **294** (2005) 43–51.

[13] C. J. Colbourn and J. H. Dinitz, *Handbook of Combinatorial Designs* (2nd edition), CRC Press, Boca Raton, 2007.

[14] A. Ekert and C. Macchiavello, Quantum error correction for communication, *Physical Review Letters*, **77**, (12) (1996) 2585–2588.

[15] M. Grassel, T. Beth and T. Pellizzari, Codes for the quantum erasure channel, *Physical Review A*, **56** (1997) 33–38.

[16] M. Harada, private communication (2010).

[17] W. Huffman and V. Pless, *Fundamentals of Error-Correcting Codes*, Cambridge University Pless, 2003.

[18] M. Jimbo and K. Shiromoto, A construction of mutually disjoint Steiner systems from isomorphic Golay codes, *J. Combin. Theory Ser. A*, **116** (2009) 1245–1251.

[19] O. Kern and G. Alber, Suppressing decoherence of quantum algorithms by jump codes, *European Physical J. D*, **36** (2005) 241–248.

[20] G.B. Khosrovshahi and B. Tayfeh-Rezaie, Large sets of t-designs through partitionable sets: A survey, *Discrete Math.*, **306** (2006) 2993-3004.

[21] E. S. Kramer and S. S. Magliveras, Some mutually disjoint Steiner systems, *J. Combin. Theory, Ser. A* **17** (1974), 39–43.

[22] F. J. MacWilliams and N. J. Sloane, *The Theory of Error-Correcting Codes*, North-Holland Publishing Company, Amsterdam, 1978.

[23] E. Knill and R. Laflamme, Theory of quantum error-correcting codes, *Physical Review A*, **55** (1997), 900-911.

[24] The Magma Computational Algebra System for Algebra, Number Theory and Geometry, Version 2.12, University of Sydney (2005).

[25] A. Munemasa, Flag-transitive 2-designs arising from line-spreads in PG($2n$-1,2), *Geometriae Dedicata*, **77** (1999), 209–213.

[26] D. Raghavarao, *Constructions and Combinatorial Problems in Design of Experiments*, Wiley, New York (1971).

[27] K. Shiromoto, private communication (2008).

[28] P. W. Shor, Scheme for reducing decoherence in quantum computer memory, *Physical Review A*, **52** (1995) R2493-R2496.

[29] A. Steane, Multiple particle interference and quantum error correction, *Proceedings of the Royal Society London Series A*, **452** (1996) 2551–2577.

[30] A. M. Steane, Error correcting codes in quantum theory, *Physical Review Letters*, **77**, (5) (1996) 793–797.

[31] L. Tierlinck, Large sets of disjoint designs and related structures, in *Contemporary Design Theory: A Collection of Surveys* J. H. Dinitz and D. R. Stinson eds., Wiley-Interscience Series in Discrete Mathematics and Optimization, (1992) 561–592.