

Matrices for graphs, designs and codes

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Abstract. The adjacency matrix of a graph can be interpreted as the incidence matrix of a design, or as the generator matrix of a binary code. Here these relations play a central role. We consider graphs for which the corresponding design is a (symmetric) block design or (group) divisible design. Such graphs are strongly regular (in case of a block design) or very similar to a strongly regular graph (in case of a divisible design). Many construction and properties for these kind of graphs are obtained. We also consider binary code of a strongly regular graph, work out some theory and give several examples.

Keywords. Block designs, divisible designs, strongly regular graphs, Seidel switching, Hadamard matrices, binary codes.

1. Introduction

Central in this reader is the interplay between graphs and designs. We start with a preliminary chapter on strongly regular graphs, block designs and their interplay. Then we will look at binary codes generated by the adjacency matrix of a strongly regular graph. This section is mainly based on [24]. The third part, based on [21], is devoted to a more recent development on graphs that are related to divisible designs. We introduce basic concepts as block designs, strongly regular graphs and Hadamard matrices, but we assume basic knowledge of algebra and graph theory. Some useful general references are [4,16,29,39].

2. Graphs and designs

2.1. Designs

A *block design* with parameters (v, k, λ) is a finite point set \mathcal{P} of cardinality v , and a collection \mathcal{B} of subsets (called *blocks*) of \mathcal{P} , such that:

- (i) Each block has cardinality k ($2 \leq k \leq v - 1$).
- (ii) Each (unordered) pair of points occurs in exactly λ blocks.

A block design with parameters (v, k, λ) is also called a 2 - (v, k, λ) design. The incidence matrix N of such a design is the $(0, 1)$ matrix with rows indexed by the points, and columns indexed by the blocks, such that $N_{ij} = 1$ if point i is in block j , and $N_{ij} = 0$ otherwise. The following result is a straightforward translation of the definition into matrix language. (as usual, J stands for an all-ones matrix, and $\mathbf{1}$ for an all-ones vector).

Proposition 2.1 A $(0, 1)$ matrix N is the incidence matrix of a $2-(v, k, \lambda)$ design if and only if

$$N^T \mathbf{1} = k\mathbf{1} \text{ and } NN^T = \lambda J + D,$$

for some diagonal matrix D .

Theorem 2.1 Suppose $(\mathcal{P}, \mathcal{B})$ is a $2-(v, k, \lambda)$ design with incidence matrix N , then

- (i) each point is incident with $r = \lambda(v - 1)/(k - 1)$ blocks, that is $N\mathbf{1} = r\mathbf{1}$, and $NN^T = \lambda J + (r - \lambda)I$,
- (ii) the number of blocks equals $b = vr/k$, that is N has b columns,
- (iii) $b \geq v$ with equality if and only if N^T is the incidence matrix of a $2-(v, k, \lambda)$ design.

Proof. (i): Fix a point $z \in \mathcal{P}$. By use of (ii) of the above definition, the number of pairs (x, a) with $x \in \mathcal{P}$, $x \neq z$ and $a \in \mathcal{B}$, $z \in a$ equals $\lambda(v - 1)$. On the other hand it is equal to $k - 1$ times the number of blocks containing z . Equation (ii) follows by counting the number of pairs (x, a) with $x \in \mathcal{P}$, $a \in \mathcal{B}$, $a \in a$ (that is, the number of ones in N). (iii): From (i) and Proposition 2.1 it follows that

$$R = \frac{1}{r - \lambda} N^T + \frac{\lambda}{r(r - \lambda)} J$$

is a right inverse of N . Therefore N has rank v , and hence $b \geq v$. Moreover, if $b = v$, then $r = k$, $R = N^{-1}$ and we have $I = N^{-1}N$, which leads to $N^T N = (k - \lambda)I + \lambda J$. By (i) we have $N\mathbf{1} = k\mathbf{1}$, hence N^T is the incidence matrix of a $2-(v, k, \lambda)$ design by Proposition 2.1. \square

A $2-(v, k, 1)$ design is also called a *Steiner 2-design*. A block design with $b = v$ is called *symmetric*. The *dual* of a design with incidence matrix N is the structure with incidence matrix N^T . Theorem 2.1(ii) states that the dual of a symmetric design is again a symmetric design with the same parameters. In terms of the original design, it means that any two distinct blocks intersect in the same number of points. In general, the size of the intersection of two distinct blocks can vary. If in a block design these numbers take only two values, we call the design *quasi-symmetric*. Obviously, two blocks in a Steiner 2-design cannot have more than one points in common, so it is symmetric, or quasi-symmetric. Note that if N is the incidence matrix of a $2-(v, k, \lambda)$ design $(\mathcal{P}, \mathcal{B})$, then $J - N$ represents a $2-(v, v - k, b - 2v + \lambda)$ design, called the *complement* of $(\mathcal{P}, \mathcal{B})$. Moreover, if N is symmetric (or quasi-symmetric), the so is the complement.

Many examples of block designs come from geometries over a finite field \mathbf{F}_q . For example the points and the lines in projective space of dimension n over \mathbf{F}_q give a $2-(q^n + q^{n-1} + \dots + q + 1, q + 1, 1)$ design. Because $\lambda = 1$, it is Steiner 2-design, and therefore quasi-symmetric, or symmetric. The design is a symmetric if and only if $n = 2$. Such a design is called a *projective plane* of order q . The smallest case $q = 2$ gives the famous Fano plane.

Another family of examples comes from Hadamard matrices. An $m \times m$ matrix H is a *Hadamard matrix* (of order m) if every entry is 1 or -1 , and $HH^T = mI$. In other words, $H^{-1} = \frac{1}{m}H^T$ hence $H^T H = mI$. If a row or a column of a Hadamard matrix is

multiplied by -1 , the matrix remains a Hadamard matrix. Therefore we can accomplish that the first row and column consist of ones only. If we then delete the first row and column we obtain a $(m-1) \times (m-1)$ matrix C , often called the *core* of H (with respect to the first row and column). It follows straightforwardly that a core C of a Hadamard matrix satisfies $CC^T = C^TC = mI - J$, and $C\mathbf{1} = C^T\mathbf{1} = -\mathbf{1}$. This implies that $N = \frac{1}{2}(C + J)$ satisfies $N^T\mathbf{1} = (\frac{1}{2}m - 1)\mathbf{1}$ and $NN^T = \frac{1}{4}mI + (\frac{1}{4}m - 1)J$, that is, N is the incidence matrix of a 2 - $(m-1, \frac{1}{2}m-1, \frac{1}{4}m-1)$ design (provided $m > 2$). Note that this implies that if $m > 2$, then m is divisible by 4.

A Hadamard matrix H is *regular* if H has constant row and column sum (ℓ say). From $HH^T = mI$ we get that $\ell^2 = m$, so $\ell = \pm\sqrt{m}$, and m is a square. If H is a regular Hadamard matrix, then we easily have that $N = \frac{1}{2}(H + J)$ is the incidence matrix of a symmetric 2 - $(m, (m + \ell)/2, (m + 2\ell)/4)$ design. Examples of Hadamard matrices are:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}.$$

One easily verifies that, if H_1 and H_2 are Hadamard matrices, then so is the Kronecker product $H_1 \otimes H_2$. Moreover, if H_1 and H_2 are regular, then so is $H_1 \otimes H_2$. With the above examples (note that the second one is regular) we can construct Hadamard matrices of order $m = 2^i$, and regular ones of order 4^i for $i \geq 0$. Many more constructions for Hadamard matrices and block designs are known. Some general references are [6] and [16], Chapter V.

2.2. Strongly regular graphs

A *strongly regular graph* with parameters (v, k, λ, μ) (often denoted by $SRG(v, k, \lambda, \mu)$) is a (simple undirected and loopless) graph of order v satisfying:

- (i) each vertex is adjacent to k ($1 \leq k \leq v - 2$) vertices,
- (ii) for each pair of adjacent vertices there are λ vertices adjacent to both,
- (iii) for each pair of non-adjacent vertices there are μ vertices adjacent to both.

For example, the pentagon is strongly regular with parameters $(v, k, \lambda, \mu) = (5, 2, 0, 1)$. One easily verifies that a graph Γ is strongly regular with parameters (v, k, λ, μ) if and only if its complement $\bar{\Gamma}$ is strongly regular with parameters $(v, v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda)$. The line graph of the complete graph of order m , known as the *triangular graph* $T(m)$, is strongly regular with parameters $(\frac{1}{2}m(m-1), 2(m-2), m-2, 4)$. The complement of $T(5)$ has parameters $(10, 3, 0, 1)$. This is the Petersen graph (see Figure 1).

A graph Γ satisfying condition (i) is called k -regular. The *adjacency matrix* of a graph Γ is the symmetric $(0, 1)$ matrix A indexed by the vertices of Γ , where $A_{ij} = 1$ if i is adjacent to j , and $A_{ij} = 0$ otherwise. It is well-known and easily seen that $A\mathbf{1} = k\mathbf{1}$ for a k -regular graph, in other words, the adjacency matrix of a k -regular graph has an eigenvalue k with eigenvector $\mathbf{1}$. Moreover, every other eigenvalue ρ satisfies $|\rho| \leq k$, and if Γ is connected, the multiplicity of k equals 1 (see Biggs [7]). For convenience we call an eigenvalue *restricted* if it has an eigenvector perpendicular to $\mathbf{1}$. So for a k -regular connected graph the restricted eigenvalues are the eigenvalues different from k .

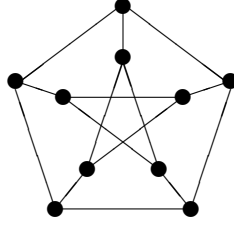


Figure 1. The Petersen graph

Theorem 2.2 For a simple graph Γ of order v , not complete or empty, with adjacency matrix A , the following are equivalent:

- (i) G is strongly regular with parameters (v, k, λ, μ) for certain integers k, λ, μ ,
- (ii) $A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J$ for certain reals k, λ, μ ,
- (iii) A has precisely two distinct restricted eigenvalues.

Proof. The equation in (ii) can be rewritten as

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Now (i) \Leftrightarrow (ii) is obvious.

(ii) \Rightarrow (iii): Let ρ be a restricted eigenvalue, and u a corresponding eigenvector perpendicular to $\mathbf{1}$. Then $Ju = 0$. Multiplying the equation in (ii) on the right by u yields $\rho^2 = (\lambda - \mu)\rho + (k - \mu)$. This quadratic equation in ρ has two distinct solutions. (Indeed, $(\lambda - \mu)^2 = 4(\mu - k)$ is impossible since $\mu \leq k$ and $\lambda \leq k - 1$.)

(iii) \Rightarrow (ii): Let r and s be the restricted eigenvalues. Then $(A - rI)(A - sI) = \alpha J$ for some real number α . So A^2 is a linear combination of A, I and J . \square

As an application, we show that quasi-symmetric block designs give rise to strongly regular graphs. Recall that a quasi-symmetric design is a 2 -(v, k, λ) design in which any two distinct blocks meet in either x or y points, for certain fixed x, y . Given this situation, we may define a graph Γ on the set of blocks, and call two blocks adjacent when they meet in x points. Then there exist coefficients $\alpha_1, \dots, \alpha_7$ such that $NN^T = \alpha_1 I + \alpha_2 J$, $NJ = \alpha_3 J$, $JN = \alpha_4 J$, $A = \alpha_5 N^T N + \alpha_6 I + \alpha_7 J$, where A is the adjacency matrix of the graph Γ . (The α_i can be readily expressed in terms of v, k, λ, x, y .) Then Γ is strongly regular by (ii) of the previous theorem. Indeed, from the equations just given it follows straightforwardly that A^2 can be expressed as a linear combination of A, I and J . We know that all 2 -($v, k, 1$) designs are quasi-symmetric. This leads to a substantial family of strongly regular graphs, including the triangular graphs $T(m)$ (derived from the trivial design consisting of all pairs out of an m -set).

Theorem 2.3 Let Γ be a strongly regular graph with adjacency matrix A and parameters (v, k, λ, μ) . Let r and s ($r > s$) be the restricted eigenvalues of A and let f and g be their respective multiplicities. Then

- (i) $k(k - 1 - \lambda) = \mu(v - k - 1)$,
- (ii) $rs = \mu - k, r + s = \lambda - \mu$,
- (iii) $f, g = \frac{1}{2}(v - 1 \mp \frac{(r+s)(v-1)+2k}{r-s})$.

(iv) r and s are integers, except perhaps when $f = g$, $(v, k, \lambda, \mu) = (4t + 1, 2t, t - 1, t)$ for some integer t .

Proof. (i) Fix a vertex x of Γ . Let $\Gamma(x)$ and $\Delta(x)$ be the sets of vertices adjacent and non-adjacent to x , respectively. Counting in two ways the number of edges between $\Gamma(x)$ and $\Delta(x)$ yields (i). The equations (ii) are direct consequences of Theorem 2.2(ii), as we saw in the proof. Formula (iii) follows from $f + g = v - 1$ and $0 = \text{trace } A = k + fr + gs = k + \frac{1}{2}(r + s)(f + g) + \frac{1}{2}(r - s)(f - g)$. Finally, when $f \neq g$ then one can solve for r and s in (iii) (using (ii)) and find that r and s are rational, and hence integral. But $f = g$ implies $(\mu - \lambda)(v - 1) = 2k$, which is possible only for $\mu - \lambda = 1$, $v = 2k + 1$. \square

These relations imply restrictions for the possible values of the parameters. Clearly, the right hand sides of (iii) must be positive integers. These are the so-called *rationality conditions*. As an example of the application of the rationality conditions we can derive the following result due to Hoffman & Singleton [27]

Theorem 2.4 Suppose $(v, k, 0, 1)$ is the parameter set of a strongly regular graph. Then $(v, k) = (5, 2), (10, 3), (50, 7)$ or $(3250, 57)$.

Proof. The rationality conditions imply that either $f = g$, which leads to $(v, k) = (5, 2)$, or $r - s$ is an integer dividing $(r + s)(v - 1) + 2k$. By use of Theorem 1(i)-(ii) we have

$$s = -r - 1, k = r^2 + r + 1, v = r^4 + 2r^3 + 3r^2 + 2r + 2,$$

and thus we obtain $r = 1, 2$ or 7 . \square

The first three possibilities are uniquely realized by the pentagon, the Petersen graph and the Hoffman-Singleton graph. For the last case existence is unknown

Except for the rationality conditions, a few other restrictions on the parameters are known. We mention two of them. The Krein conditions [35], can be stated as follows:

$$(r + 1)(k + r + 2rs) \leq (k + r)(s + 1)^2,$$

$$(s + 1)(k + s + 2rs) \leq (k + s)(r + 1)^2.$$

The absolute bound (see Delsarte, Goethals & Seidel [17]) reads,

$$v \leq f(f + 3)/2, v \leq g(g + 3)/2.$$

The Krein conditions and the absolute bound are special cases of general inequalities for association schemes, see for example [10]. For constructions and more results on strongly regular graphs we refer to [11], [12], [15], [16], [28], or [36].

2.3. Neighborhood designs

Any graph Γ can be interpreted as a design, by taking the vertices of Γ as points, and the neighborhoods of the vertices as blocks. In other words, the adjacency matrix of Γ is interpreted as the incidence matrix of a design. Let us call such a design the *neighborhood design* of Γ .

Consider a strongly regular graph Γ with parameters (v, k, λ, μ) . If $\lambda = \mu$, then any two distinct vertices have exactly λ common neighbors, and the adjacency matrix A of Γ satisfies

$$AA^\top = A^2 = (k - \lambda)I + \lambda J.$$

This implies that the neighborhood design of Γ is a symmetric 2 - (v, k, λ) design (sometimes called: (v, k, λ) design). Rudvalis [34] has called such a graph a (v, k, λ) graph. If a symmetric design admits a symmetric incidence matrix, the corresponding bijection between points and blocks is called a *polarity* of the design. The points (and blocks) that correspond to a 1 on the diagonal are the absolute points (blocks) of the polarity. Thus a (v, k, λ) design with a polarity with no absolute points can be interpreted as a (v, k, λ) graph.

Similarly, if A is the adjacency matrix of a strongly regular graph with parameters $(v, k, \lambda, \lambda + 2)$, then $A + I$ is the incidence matrix of a square 2 - (v, k, λ) design, and in this way one obtains precisely the 2 - (v, k, λ) designs possessing a polarity with all points absolute.

This interplay between graphs and designs turned out to be fruitful for both parts. For example, an easy construction of a symmetric 2 - $(16, 6, 2)$ design goes via the 4×4 grid, (that is, the line graph of the complete bipartite graph $K_{4,4}$, also known as the Lattice graph $L(4)$), which is a $(16, 6, 2)$ graph. It may happen, however, that two non-isomorphic (v, k, λ) graphs, Γ_1 and Γ_2 with adjacency matrices A_1 and A_2 say, give isomorphic designs. Also A_1 and $A_2 + I$ can represent isomorphic designs. The standard example is given by the two $SRG(16, 6, 2, 2)$'s (the lattice graph $L(4)$ and the Shrikhande graph) and the unique $SRG(16, 5, 0, 2)$ (the Clebsch graph). The three graphs produce the same symmetric 2 - $(16, 6, 2)$ design.

Proposition 2.2 *If two non-isomorphic (v, k, λ) graphs Γ_1 and Γ_2 give rise to isomorphic (v, k, λ) designs, then both Γ_1 and Γ_2 have an involution (that is, an automorphism of order 2).*

Proof. Let A_i be the adjacency matrix of Γ_i ($i = 1, 2$), and assume that the corresponding designs are isomorphic. Then there exist permutation matrices P and Q such that $PA_1Q = A_2$. Without loss of generality we assume $Q = I$ (otherwise replace A_2 by $Q^\top A_2 Q$). The symmetry of A_2 gives $PA_1 = A_1 P^\top$, and hence $P^m A_1 = A_1 (P^m)^\top$. If P has even order $2m$, then $P^{2m} = I$ and $P^m = (P^m)^\top \neq I$. This implies $A_1 = P^{2m} A_1 = P^m A_1 (P^m)^\top$, so P^m is an involution. If P has odd order $2m - 1$, then $A_2 = PA_1 = P^{2m} A_1 = P^m A_1 (P^m)^\top$, so Γ_1 and Γ_2 are isomorphic graphs. \square

So, if for example a (v, k, λ) graph Γ has a trivial automorphism group, then any other (v, k, λ) graph not isomorphic to Γ gives a non-isomorphic design. For instance, there exist 16428 $(36, 21, 12)$ graphs. From these graphs, 15127 have a trivial automorphism group (see [37], [30]). So at least 15128 are also non-isomorphic as designs.

A large family of (v, k, λ) graphs comes from regular graphical Hadamard matrices. A Hadamard matrix H is graphical if it is symmetric with constant diagonal. Without loss of generality we assume that the diagonal elements are -1 (otherwise we replace H by $-H$). If, in addition, H is regular of order m with row sum $\ell = \pm\sqrt{m}$, then $A = \frac{1}{2}(H + J)$ is the adjacency matrix of an $(m, (m + \ell)/2, (m + 2\ell)/4)$ graph. The two smallest regular graphical Hadamard are:

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}.$$

It is easily verified that if H_1 and H_2 are regular graphical Hadamard matrices with row sums ℓ_1 and ℓ_2 , respectively, then the Kronecker product $H_1 \otimes H_2$ is again such a matrix, whose row sum is $\ell_1 \ell_2$. Starting with the above Hadamard matrices, we can make regular graphical Hadamard matrices of order $m = 4^t$ with row sum $\ell = 2^t$ and $\ell = -2^t$. Many more constructions are known, for example if $m = 4t^4$, $t \geq 1$ for $\ell = 2t^2$ and $\ell = -2t^2$ (see [16] for a survey, and [26] for some recent developments).

3. Binary codes of strongly regular graphs

3.1. Introduction

Codes generated by the incidence matrix of combinatorial designs and related structures have been studied rather extensively. The best reference for this is the book by Assmus and Key [4] (see also the update [5]). Codes generated by the adjacency matrix of a graph did get less attention. For strongly regular graphs there is much analogy with designs and therefore interesting results may be expected. Concerning the dimension of these codes, that is, the p -rank of strongly regular graphs, several results are known: see [9], [33]. It has turned out that some special strongly regular graphs generate nice codes, see [23] and [38]. Here we restrict to binary codes, not only because it is the simplest case, but also since for the binary case there is a relation with regular two-graphs and Seidel switching that has already proved to be useful: see [23] and [14].

For an integral $n \times v$ matrix A we define the binary code \mathcal{C}_A of A to be the subspace of $\mathbf{V} = \mathbf{F}_2^v$ generated by the rows of $A \pmod{2}$. We start with some known lemmas for symmetric integral matrices (see [9], [13] or [33]).

Lemma 3.1 *If A is a symmetric integral matrix with zero diagonal, then $2\text{-rank}(A)$ (i.e. the dimension of \mathcal{C}_A) is even.*

Proof. Let A' be a non-singular principal submatrix of A with the same 2-rank as A . Over \mathbf{Z} , any skew symmetric matrix of odd order has determinant 0 (since $\det(A) = -\det(A^\top)$). Reduction mod 2 shows that A' has even order. \square

Lemma 3.2 *If A is a symmetric binary matrix, then $\text{diag}(A) \in \mathcal{C}_A$.*

Proof. Suppose $x \in \mathcal{C}_A^\perp$. Then $\sum_i (A)_{ii} x_i = \sum_{i,j} (A)_{ij} x_i x_j = x^\top A x = 0 \pmod{2}$, so $x \perp \text{diag}(A)$. Hence $\text{diag}(A) \in \mathcal{C}_A$. \square

With these lemmas we easily find a relation between the codes \mathcal{C}_A and \mathcal{C}_{A+J} .

Proposition 3.1 *Suppose A is the adjacency matrix of a graph then $\mathcal{C}_A \subseteq \mathcal{C}_{A+J}$ and the following are equivalent:*

- (i) $\mathcal{C}_A = \mathcal{C}_{A+J}$, (ii) $\mathbf{1} \in \mathcal{C}_A$, (iii) $\dim(\mathcal{C}_{A+J})$ is even.

Proof. By Lemma 3.2, $\text{diag}(A + J) = \mathbf{1} \in \mathcal{C}_{A+J}$, so $\mathcal{C}_{A+J} = \mathcal{C}_A + \langle \mathbf{1} \rangle$ and the equivalence of (i) and (ii) follows. By Lemma 3.1 we have $2\text{-rank}(A)$ is even and so (i) \Leftrightarrow (iii). \square

The next proposition gives a trivial but useful relation between \mathcal{C}_A and \mathcal{C}_{A+I} .

Proposition 3.2 *If A is a symmetric integral matrix, then $\mathcal{C}_A^\perp \subseteq \mathcal{C}_{A+I}$ with equality if and only if $A(A + I) = 0 \pmod{2}$.*

Proof. Suppose $x \in \mathcal{C}_A^\perp$. Then $Ax = 0 \pmod{2}$, so $(A + I)x = x$ and hence $x \in \mathcal{C}_{A+I}$. Clearly $A(A + I) = 0 \pmod{2}$ reflects that $\mathcal{C}_{A+I} \subseteq \mathcal{C}_A^\perp$, which completes the proof. \square

3.2. Facts from the parameters

Here we present some properties of the binary codes of a strongly regular graph Γ , using only the parameters (eigenvalues) of Γ .

Proposition 3.3 *Suppose Γ has non-integral eigenvalues.*

- (i) *If μ is odd (i.e. $v = 5 \pmod{8}$) then $\mathcal{C}_A = \mathbf{1}^\perp$ and $\mathcal{C}_{A+I} = \mathbf{V}$.*
- (ii) *If μ is even ($v = 1 \pmod{8}$) then $\mathcal{C}_A^\perp = \mathcal{C}_{A+I}$ and $\dim(\mathcal{C}_A) = \dim(\mathcal{C}_{A+I}) - 1 = 2\mu (= f = g = k = (v - 1)/2)$.*

Proof. If μ is odd, Equation 2.2(ii) becomes $A^2 = A + I + J \pmod{2}$, so $(A + J)(A + I) = I \pmod{2}$, hence $\mathcal{C}_{A+J} = \mathcal{C}_{A+I} = \mathbf{V}$ and $\mathcal{C}_A = \mathbf{1}^\perp$. Suppose μ is even. Then $A^2 = A \pmod{2}$ so $\mathcal{C}_{A+I} = \mathcal{C}_A^\perp$. The characteristic polynomial of A is given by:

$$\det(xI - A) = (x + k)(x^2 + x + \mu)^f = x^{f+1}(x + 1)^f \pmod{2}.$$

Therefore $2\text{-rank}(A + I) \geq v - f$ and $2\text{-rank}(A) \geq v - (f + 1) = f$. We know (Proposition 3.2) $2\text{-rank}(A) + 2\text{-rank}(A + I) = v$, and the result follows. \square

Proposition 3.4 *Suppose the eigenvalues r and s of Γ are integers.*

- (i) *If $k = r = s = 1 \pmod{2}$ then $\mathcal{C}_A = \mathbf{V}$, \mathcal{C}_{A+I} is self-orthogonal and $\dim(\mathcal{C}_{A+I}) \leq \min\{f + 1, g + 1\}$.*
- (ii) *If $r = s = 1 \pmod{2}$ and k is even, then $\mathcal{C}_A = \mathbf{1}^\perp$, \mathcal{C}_{A+I} is orthogonal to \mathcal{C}_A and $\dim(\mathcal{C}_{A+I}) \leq \min\{f + 1, g + 1\}$.*
- (iii) *If $r \neq s \pmod{2}$ and k is even, then $\mathcal{C}_{A+I} = \mathcal{C}_A^\perp$, $\dim(\mathcal{C}_A) = f'$ and $\dim(\mathcal{C}_{A+I}) = v - f'$, where f' is the multiplicity of the odd eigenvalue.*
- (iv) *If $r \neq s \pmod{2}$ and k is odd, then $\mathcal{C}_A = \mathcal{C}_A^\perp$, $\dim(\mathcal{C}_A) = f' + 1$ and $\dim(\mathcal{C}_{A+I}) = v - f'$.*
- (v) *If $r = s = 0 \pmod{2}$ then k is even, $\mathcal{C}_{A+I} = \mathbf{V}$, \mathcal{C}_A is self-orthogonal and $\dim(\mathcal{C}_A) \leq \min\{f + 1, g + 1\}$ and even.*

Proof. (i): Equation 2.2(ii) gives $A^2 = I$ and $(A + I)^2 = 0 \pmod{2}$. Over the real numbers, $\text{rank}(A - rI) = v - f = g + 1$, hence $2\text{-rank}(A + I) \leq g + 1$ and similarly, $2\text{-rank}(A + I) \leq f + 1$.

(ii): Now $A\mathbf{1} = 0$, $A^2 = I + J$, and $(A + I)^2 = J \pmod{2}$, proving the first two claims. For the dimension bound see case (i).

(iii): Now Equation 2.2(ii) becomes $A(A + I) = 0 \pmod{2}$, so $\mathcal{C}_{A+I} = \mathcal{C}_A^\perp$ by Proposition 3.2. The characteristic polynomial of $A \pmod{2}$ reads $x^{v-f'}(x+1)^{f'}$, so $\dim(\mathcal{C}_{A+I}) \geq v - f'$ and $\dim(\mathcal{C}_A) \geq f'$ and, since they add up to v the result follows.
(iv): Here $A\bar{A} = 0 \pmod{2}$. Similar to case (iii) we get $\dim(\mathcal{C}_{A+I}) \geq v - f' - 1$ and $\dim(\mathcal{C}_A) \geq f' + 1$. Now the dimensions add up to $v + 1$, but f' is odd (from $\text{trace}(A)$) and v is even (since k is odd), so by Proposition 3.1 we find $\dim(\mathcal{C}_A) = f' + 1$, $\dim(\mathcal{C}_{A+I}) = v - f'$ and $\dim(\mathcal{C}_{\bar{A}}) = v - f' - 1$.
(v): Now $A^2 = kJ$ and $(A + I)^2 = kJ + I \pmod{2}$. From $k + fr + gs = 0$ it follows that k is even. By Lemma 3.1 $\dim(\mathcal{C}_A)$ is even. The rest follows by similar arguments as above. \square

Thus, unless r and s are both even, the dimension of \mathcal{C}_A (i.e. $2\text{-rank}(A)$) follows from the parameters of Γ and similarly, $\dim(\mathcal{C}_{A+I})$ follows, unless r and s are both odd (see [9]). From the two propositions above we also see that if $rs (= \mu - k)$ is odd \mathcal{C}_A and \mathcal{C}_{A+I} ($= \mathcal{C}_{\bar{A}+I}$) are determined by the parameters of Γ . Similarly, \mathcal{C}_{A+I} and $\mathcal{C}_{\bar{A}}$ are determined if $(r+1)(s+1)$ is odd. So in these cases non-isomorphic strongly regular graphs with the same parameters (of which there are many examples) generate the same (trivial) codes.

3.3. Some families and their codes

3.3.1. Triangular graphs

The triangular graph $T(n)$ is the line graph of the complete graph K_n . It follows that $T(n)$ is a strongly regular graph with $v = n(n-1)/2$, $k = 2(n-2)$, $\lambda = n-2$, $\mu = 4$, $r = n-4$ and $s = -2$. $T(n)$ is known to be determined by these parameters if $n \neq 8$. If N is the vertex-edge incidence matrix of K_n , then $A = N^\top N \pmod{2}$ is the adjacency matrix of $T(n)$. The words of \mathcal{C}_N , \mathcal{C}_A and \mathcal{C}_{A+I} are characteristic vectors of subsets of the edge set of K_n , so can be interpreted as graphs on a fixed vertex set of size n . It is easily seen that \mathcal{C}_N is the $n-1$ dimensional binary code consisting of all complete bipartite graphs and that \mathcal{C}_N^\perp consists of disjoint unions of Euler graphs. Note that $\mathbf{1} \notin \mathcal{C}_N$.

Theorem 3.1 *Let Γ be the triangular graph $T(n)$.*

If n is even then $\mathcal{C}_A = \mathcal{C}_N \cap \mathbf{1}^\perp$ (the Eulerian complete bipartite graphs), $\mathcal{C}_{A+I} = \mathbf{V}$, $\mathcal{C}_{\bar{A}} = \mathbf{V}$ if $n \equiv 0 \pmod{4}$ and $\mathcal{C}_{\bar{A}} = \mathbf{1}^\perp$ if $n \equiv 2 \pmod{4}$.

If n is odd then $\mathcal{C}_A = \mathcal{C}_N$, $\mathcal{C}_{A+I} = \mathcal{C}_N^\perp$, $\mathcal{C}_{\bar{A}} = \mathcal{C}_N^\perp$ if $n \equiv 1 \pmod{4}$ and $\mathcal{C}_{\bar{A}} = \mathcal{C}_N^\perp \cap \mathbf{1}^\perp$ (the unions of Euler graphs with an even total number of edges) if $n \equiv 3 \pmod{4}$.

Proof. Since $N^\top N = A \pmod{2}$, we have $\mathcal{C}_A \subset \mathcal{C}_N$. First suppose n is odd. By *iii* of Proposition 3.4, $\dim(\mathcal{C}_A) = f = n-1$, hence $\mathcal{C}_A = \mathcal{C}_N$ and $\mathcal{C}_{A+I} = \mathcal{C}_N^\perp$. Proposition 3.1 gives $\mathcal{C}_{\bar{A}} = \mathcal{C}_{A+I}$ whenever $(n-1)(n-2)/2 = \dim(\mathcal{C}_{A+I})$ is even, that is $n \equiv 1 \pmod{4}$. If $n \equiv 3 \pmod{4}$, $\mathcal{C}_{\bar{A}}$ has dimension one less and is orthogonal to \mathcal{C}_A and to $\mathbf{1}$. Since $\mathbf{1} \notin \mathcal{C}_A$, this proves the last claim. Next take n even. By *i* and *ii* of Proposition 3.4 we find \mathcal{C}_{A+I} and $\mathcal{C}_{\bar{A}}$. Since $\dim(\text{kernel}(N^\top)) \equiv 1 \pmod{2}$, $\dim(\mathcal{C}_A) \geq \dim(\mathcal{C}_N) - 1 = n-2$. Clearly $\mathbf{1} \in \mathcal{C}_A^\perp$ but (since n is even), $\mathbf{1} \notin \mathcal{C}_N^\perp$. Therefore $\mathcal{C}_A^\perp = \mathcal{C}_N^\perp + \langle \mathbf{1} \rangle$ and so $\mathcal{C}_A = \mathcal{C}_N \cap \mathbf{1}^\perp$. \square

From Theorem 3.1 it follows that the codes \mathcal{C}_N and \mathcal{C}_A only have weights $w_i = i(n-i)$

($0 \leq i \leq \frac{n}{2}$). In n is odd, the number of codewords of weight w_i equals $\binom{n}{i}$ (for both \mathcal{C}_N and \mathcal{C}_A). If n is even, \mathcal{C}_N has $\binom{n}{i}$ codewords of weight w_i for $0 \leq i < \frac{n}{2}$ and $\frac{1}{2}\binom{n}{n/2}$ codewords of weight $w_{n/2}$. The code \mathcal{C}_A consists of the codewords from \mathcal{C}_N with even weight.

3.3.2. Lattice graphs

The lattice graph $L(m)$ is the line graph of the complete bipartite graph $K_{m,m}$. It is strongly regular with parameters $v = m^2$, $k = 2(m-1)$, $\lambda = m-2$, $\mu = 2$, $r = n-2$ and $s = -2$. If $m \neq 4$, $L(m)$ is determined by these parameters. Similar to above the adjacency matrix $A = M^\top M \pmod{2}$ if M is the vertex-edge incidence matrix of $K_{m,m}$. The code \mathcal{C}_M^\perp consist of the edge sets of $K_{m,m}$ that form a union of Euler graphs. The code \mathcal{C}_M has dimension $2m-1$ and consists of disjoint unions of two bipartite graphs, one on m_1+m_2 and one on $(m-m_1)+(m-m_2)$ vertices. Each choice of m_1, m_2 ($0 \leq m_1 \leq m, 0 \leq m_2 \leq m/2$) gives codewords of weight $m_1m_2+(m-m_1)(m-m_2)$. The number of these codewords equals $\binom{m}{m_1}\binom{m}{m_2}$ if $m_2 < m/2$ and $\frac{1}{2}\binom{m}{m_1}\binom{m}{m/2}$ if $m_2 = m/2$ (but note that different choices for m_1, m_2 can lead to the same weight). The weight enumerators of the codes \mathcal{C}_A now follow easily from the next result.

Theorem 3.2 *Let Γ be the lattice graph $L(m)$.*

If m is even then \mathcal{C}_A consists of the graphs from \mathcal{C}_M with $m_1 + m_2$ odd, and moreover, $\mathcal{C}_A + \langle \mathbf{1} \rangle = \mathcal{C}_M$ and $\mathcal{C}_{A+I} = \mathcal{C}_A^\perp = \mathbf{V}$.

If m is odd then \mathcal{C}_A consists of the graphs from \mathcal{C}_M with $m_1 + m_2$ even, and moreover, $\mathcal{C}_A = \mathcal{C}_M \cap \mathbf{1}^\perp$, $\mathcal{C}_{A+I} = \mathcal{C}_A^\perp$ and $\mathcal{C}_A^\perp = \mathcal{C}_{A+I} \cap \mathbf{1}^\perp$.

Proof. From $M^\top M = A \pmod{2}$, we deduce $\mathcal{C}_A \subseteq \mathcal{C}_M$ and $\dim(\mathcal{C}_A) \geq \dim(\mathcal{C}_M) - 1 = 2m - 2$. Let $\chi \in \mathbf{F}_2^v$ represent a subgraph of $K_{m,m}$ with all vertex degrees odd (if m is odd, we may choose $\chi = \mathbf{1}$). Then $\chi \in \mathcal{C}_A^\perp$, but $\chi \notin \mathcal{C}_M^\perp$, hence $\mathcal{C}_A = \mathcal{C}_M \cap \chi^\perp$. Now all statements follow straightforwardly. \square

3.3.3. Paley graphs

Suppose $v = 1 \pmod{4}$ is a prime power. The *Paley graph* has \mathbf{F}_v as vertex set and two vertices are adjacent if the difference is a non-zero square in \mathbf{F}_v . The Paley graph is an $SRG(v, (v-1)/2, (v-1)/4 - 1, (v-1)/4)$ which is isomorphic to its complement. By Propositions 3.3 and 3.4, the code \mathcal{C}_A of a Paley graph is only non-trivial if $v = 1 \pmod{8}$. Then \mathcal{C}_A and \mathcal{C}_{A+I} are well known as the (binary) quadratic residue codes, see for example [15] or [29] (which are usually only defined for primes v). For $v = 5, 9, 13$ and 17 , the Paley graph is the only one with the given parameters. If $v \geq 25$, other graphs with the same parameters exist. If $v = 5 \pmod{8}$ all these graphs give isomorphic (trivial) codes. If $v = 25$ or 41 (see Section 3.5), the known non-isomorphic graphs give non-isomorphic codes and amongst them, the codes of the Paley graphs have the largest minimum distance. We conjecture that the second part of this statement is true in general.

3.3.4. Graphs from designs and Latin squares

Let D denote a $2-(n, \kappa, 1)$ design with incidence matrix N . Then $A = N^\top N - \kappa I$ is the adjacency matrix of a strongly regular graph Γ_D with parameters $(m^2 - m(m-1)/\kappa, \kappa(m-1), \kappa^2 - 2\kappa + m - 1, \kappa^2)$, where $m = (n-1)(\kappa-1)$. We have $\mathcal{C}_A =$

$\mathcal{C}_{N^\top N} \subseteq \mathcal{C}_N$ if κ is even and $\mathcal{C}_{A+I} = \mathcal{C}_{N^\top N} \subseteq \mathcal{C}_N$ if κ is odd. If $\kappa = 2$, $\Gamma(D)$ is a triangular graph and the related codes are given above. If $\kappa = 3$ D is a Steiner triple system $STS(n)$.

A Latin square of order m (denoted by $LS(m)$) is an $m \times m$ matrix L with entries from $\{1, \dots, m\}$ such that every entry occurs exactly once in every row and column. A Latin square can be represented by a set of m^2 triples (i, j, k) indicating that entry (i, j) is equal to k . Then two triples of at most one entry in common. The Latin square graph Γ_L of L is defined on the triples (the entries of L), where two triples are adjacent if they have an element in common (that is, the entries are in the same row, the same column, or have the same value). Then it easily follows that Γ_L is an $SRG(m^2, \kappa(m-1), \kappa^2 - 3\kappa + m, \kappa(\kappa-1))$. Let N be the $3m \times m^2$ incidence of this the set of triples of a L . Then we easily have that $A = N^\top N - 3I$ is the adjacency matrix of Γ_L , and $\mathcal{C}_{A+I} = \mathcal{C}_{N^\top N} \subseteq \mathcal{C}_N$.

For Γ_D and Γ_L , the dimensions of \mathcal{C}_N and \mathcal{C}_{A+I} are known in terms of the number of sub-triple systems and quotient Latin squares, see [18], [31] and [33]. In some cases the relation between \mathcal{C}_N and \mathcal{C}_{A+I} is easy.

Proposition 3.5 *If D is an $STS(n)$ then*

- (i) *if $n \equiv 1 \pmod{4}$ (i.e. m is even), then $\mathcal{C}_{A+I} = \mathcal{C}_N$ and $\dim(\mathcal{C}_{A+I}) = n$;*
- (ii) *if $n \equiv 3 \pmod{4}$ (i.e. m is odd), then $\dim(\mathcal{C}_{A+I}) = 2\dim(\mathcal{C}_N) - n$ (so $\mathcal{C}_{A+I} = \mathcal{C}_N$ if and only if $\dim(\mathcal{C}_N) = n$).*

If D represents an $LS(m)$ then $\dim(\mathcal{C}_N) \leq 3m - 2$ and

- (iii) *if m is odd then $\mathcal{C}_{A+I} = \mathcal{C}_N$ and $\dim(\mathcal{C}_{A+I}) = 3m - 2$;*
- (iv) *if m is even then $\dim(\mathcal{C}_{A+I}) \leq 3m - 4$ with equality if and only if $\dim(\mathcal{C}_N) = 3m - 2$; equality also implies that $\mathcal{C}_{A+I} = \mathcal{C}_N \cap \mathcal{C}_N^\perp$.*

Proof. The cases (i) and (iii) follow from Proposition 3.4 and the results about dimensions in (ii) and (iv) can be found in Chapter 3 of [33]. So we are left with the last statement. We have $NN^\top = (J_3 + I_3) \otimes J_m \pmod{2}$ and $\dim(\mathcal{C}_N \cap \mathcal{C}_N^\perp) \leq \dim(\mathcal{C}_N) - 2\text{-rank}(NN^\top) = 3m - 4$. Moreover, $NN^\top N = 0$, so $\mathcal{C}_{A+I} \perp \mathcal{C}_N$ and hence $\mathcal{C}_{A+I} \subseteq \mathcal{C}_N \cap \mathcal{C}_N^\perp$ and the result follows. \square

For Steiner triple systems the problem has been raised (see [38]) whether or not non-isomorphic designs always give non-isomorphic codes \mathcal{C}_N . This is true for $n \leq 15$. If $\dim(\mathcal{C}_{A+I}) < n$ (the $STS(n)$ has subsystems) then $\mathcal{C}_{A+I} \neq \mathcal{C}_N$. also the codes \mathcal{C}_{A+I} are mutually non-isomorphic. However, there exist examples of non-isomorphic strongly regular graphs with the parameters of the graph of an $STS(15)$, but with isomorphic codes \mathcal{C}_{A+I} of dimension 15 (see [24]).

The binary codes of Latin squares have also been studied by Assmus [3]. he wonders if non-isomorphic Latin squares (regarded as nets of degree 3) give non-isomorphic codes \mathcal{C}_N . This is true for $m \leq 7$. In particular if $m = 4$ the codes \mathcal{C}_N of the two Latin squares even have different dimension. However the codes \mathcal{C}_{A+I} of the graphs are isomorphic, because they correspond to the same 2-(16, 10, 6) design (see the end of Section 2.3).

3.4. Two-graph codes

We briefly explain Seidel switching. For details we refer to [11] or [15]. Let $\Gamma = (V, E)$ be a graph and let $\{V_1, V \setminus V_1\}$ be a partition of V , then we define the result of switching

Γ with respect to this partition to be the graph $\Gamma' = (V, E')$ whose edges are those edges of Γ contained in V_1 or $V \setminus V_1$ together with the pairs $\{v_1, v_2\}$, with $v_1 \in V_1, v_2 \in V \setminus V_1$ for which $\{v_1, v_2\} \notin E$. The graphs Γ and Γ' are said to be switching equivalent. It is not hard to check that switching defines an equivalence relation on graphs. An equivalence class is called a two-graph. Note that, if we switch with respect to the set of neighbors Γ_x of a vertex x , then x becomes an isolated vertex in Γ' . If we order the vertices in a suitable way then, in terms of the adjacency matrices A and A' , Seidel switching comes down to

$$A = \begin{bmatrix} A_1 & A_{12} \\ A_{12}^\top & A_2 \end{bmatrix}, A' = \begin{bmatrix} A_1 & A_{12} + J \\ A_{12}^\top + J & A_2 \end{bmatrix} \pmod{2}.$$

Suppose we switch with respect to a subset V_1 of V with characteristic vector χ . Then we have

$$\mathcal{C}_A + \langle \mathbf{1} \rangle + \langle \chi \rangle = \mathcal{C}_{A'} + \langle \mathbf{1} \rangle + \langle \chi \rangle.$$

Let us not worry about $\mathbf{1}$ and look at the codes $\mathcal{C}_{A+J} = \mathcal{C}_A + \langle \mathbf{1} \rangle$ and $\mathcal{C}_{A'+J}$. It is clear that if $\chi \in \mathcal{C}_{A+J}$ then $\mathcal{C}_{A'+J} \subseteq \mathcal{C}_{A+J}$. Suppose Γ and Γ' both have an isolated vertex (not the same one) then χ is in \mathcal{C}_{A+J} and $\mathcal{C}_{A'+J}$, hence $\mathcal{C}_{A+J} = \mathcal{C}_{A'+J}$. So this code is independent of the isolated vertex and we will call it the *two-graph code*. Note that $\mathbf{1} \notin \mathcal{C}_A$ (because of the isolated vertex), so $\dim(\mathcal{C}_{A+J}) = \dim(\mathcal{C}_A) + 1$ is odd.

Assume Γ is an $SRG(v, k, \lambda, \mu)$ with $k = 2\mu$ (or equivalently, $k = -2rs$). Extend Γ with an isolated vertex x to $\tilde{\Gamma}$ (i.e. $\tilde{\Gamma} \setminus \{x\} = \Gamma$). If we switch in $\tilde{\Gamma}$ to $\tilde{\Gamma}'$, such that another vertex y becomes isolated, then it follows that $\Gamma' = \tilde{\Gamma}' \setminus \{y\}$ is again a $SRG(v, k, \lambda, \mu)$, but not necessarily isomorphic to Γ . In this case the switching class of $\tilde{\Gamma}$ is called a *regular two-graph* and Γ (and Γ') is the *descendant* of $\tilde{\Gamma}$ with respect to x (and y). Clearly, the code \mathcal{C}_A of a descendant is the shortened code of the corresponding two-graph code. Regular two-graphs can produce interesting two-graph codes. For example the Paley graph is the descendant of a regular two-graph and the corresponding two-graph code is the extended quadratic residue code. For other interesting two-graph codes, see [14], [22] and [23]. If $\tilde{\Gamma}$ can be switched into a regular graph $\tilde{\Gamma}'$, then it follows that $\tilde{\Gamma}'$ is strongly regular with the same r and s as Γ , but with two possibilities for the valency: Either $k = -2rs - r$ or $k = -2rs - s$ (so r and s need to be integral). On the other hand, a strongly regular graph with degree $-2rs - r$ or $-2rs - s$ is in the switching class of a regular two-graph (so isolating a vertex yields a strongly regular graph with $k = -2rs$). For example the Shrikhande graph, $L(4)$ and the complement of the Clebsch graph are switching equivalent. We observed already that these three graphs generate the same (6-dimensional) code. By isolating a vertex we get $T(6)$ and the two-graph code is a 5-dimensional subspace of the $L(4)$ code. The shortened code (with respect to any vertex) is the 4-dimensional code of $T(6)$.

Theorem 3.3 *Suppose Ω is a regular two-graph with eigenvalues r and s and two-graph code \mathcal{C} . Suppose Γ is a k -regular graph in Ω (so Γ is strongly regular) and let Δ be the graph in Ω with a given vertex x isolated (so switching in Γ with respect to the neighbors Γ_x of x gives Δ). Let A and B be the adjacency matrices of Γ and Δ respectively, and let χ denote the characteristic vector of the switching set Γ_x . Then either*

$$\left. \begin{array}{l} \dim \mathcal{C}_A = \dim \mathcal{C}_B = \dim \mathcal{C} - 1 \\ \mathbf{1} \notin \mathcal{C}_A \\ \chi \in \mathcal{C}_B \\ \mathcal{C}_{A+J} = \mathcal{C}_A + \langle \mathbf{1} \rangle = \mathcal{C}_B + \langle \mathbf{1} \rangle = \mathcal{C} \end{array} \right\} \text{or} \left\{ \begin{array}{l} \dim \mathcal{C}_A - 2 = \dim \mathcal{C}_B = \dim \mathcal{C} - 1 \\ \mathbf{1} \in \mathcal{C}_A \\ \chi \notin \mathcal{C}_B \\ \mathcal{C}_{A+J} = \mathcal{C}_A = \mathcal{C}_B + \langle \mathbf{1} \rangle + \langle \chi \rangle = \mathcal{C} + \langle \chi \rangle. \end{array} \right.$$

If k is even and $r + s$ is odd, we are in the first case.

If $k = 2 \pmod 4$ and $r + s$ is even, or k is odd, we are in the second case.

Proof. The results follow from the fact that

$$\mathcal{C}_A + \langle \mathbf{1} \rangle + \langle \chi \rangle = \mathcal{C}_B + \langle \mathbf{1} \rangle + \langle \chi \rangle, \mathcal{C}_A + \langle \mathbf{1} \rangle = \mathcal{C}_{A+J}, \mathcal{C} = \mathcal{C}_B + \langle \mathbf{1} \rangle,$$

and that $\dim \mathcal{C}_A$ and $\dim \mathcal{C}_B$ are even. Clearly $\mathbf{1} \notin \mathcal{C}_B$, $\mathbf{1} \in \mathcal{C}_{A+J}$ and $\chi \in \mathcal{C}_A$.

If $\mathbf{1} \in \mathcal{C}_A$ then $\mathcal{C}_A = \mathcal{C}_{A+J}$ and $\mathcal{C}_A = \mathcal{C}_B + \langle \mathbf{1} \rangle + \langle \chi \rangle$, so \mathcal{C}_B is a proper subspace of \mathcal{C}_A and hence $\dim \mathcal{C}_A = \dim \mathcal{C}_B + 2$ and $\chi \notin \mathcal{C}_B$. On the other hand, if $\mathbf{1} \notin \mathcal{C}_A$, then χ must be a codeword of \mathcal{C}_B and $\dim \mathcal{C}_A = \dim \mathcal{C}_B$. Furthermore, $\mathcal{C}_{A+J} = \mathcal{C}_A + \langle \mathbf{1} \rangle = \mathcal{C}_B + \langle \mathbf{1} \rangle = \mathcal{C}$.

If k is even and $r + s$ is odd, then $\mu = k + rs$ is even and $\lambda = \mu + r + s$ is odd. Now the rows of B corresponding to Γ_x add up to the characteristic vector χ of Γ_x . So $\chi \in \mathcal{C}_B$ and hence we are in the first case.

It is clear that $\mathbf{1} \in \mathcal{C}_A$ if k is odd. Suppose $k = 2 \pmod 4$ and $r + s$ is even. Then r and s are both even (since $-k = 2rs + s$ or $2rs + r$). Let B' be the adjacency matrix of the descendant $\Delta' = \Delta \setminus \{x\}$. Then $\mathcal{C}_{B'}$ is self-orthogonal by 3.4.v. Moreover, the degree of Δ' is $2rs$, which is divisible by 4, and hence all weights in $\mathcal{C}_{B'}$ and \mathcal{C}_B are divisible by 4. Therefore $\chi \notin \mathcal{C}_B$, so we are in the second case. \square

For example, the last statement implies that $\mathbf{1} \in \mathcal{C}_A$ for an $SRG(36, 14, 4, 6)$. If k is even and $r + s$ is odd then $\mathcal{C} = \mathcal{C}_{A+J}$. So, in this case, non-isomorphic switching equivalent strongly regular graphs give isomorphic codes of the form \mathcal{C}_{A+J} . Examples are given by the switching equivalent $SRG(26, 10, 3, 4)$'s (see the next section).

It is clear that if two two-graph codes are isomorphic then so are the codes of corresponding descendants. And vice versa, two descendants Γ_1 and Γ_2 with isomorphic codes $\mathcal{C}_{A_1} = \mathcal{C}_{A_2}$ give isomorphic two-graph codes. Among the regular two-graphs on 36 vertices ($r = 2, s = -4$) there exist several non-isomorphic ones with isomorphic two-graph codes, therefore we also have non-isomorphic $SRG(35, 16, 6, 8)$'s with isomorphic codes \mathcal{C}_A (see [24]).

3.5. Small cases

In Table 1 we give the parameters of all primitive strongly regular graphs on at most 40 vertices (up to taking complements). We indicate how many non-isomorphic graphs there exist with the given parameters and, if $k = 2\mu$ we give the number of corresponding non-isomorphic regular two-graphs. In the previous sections we have obtained the codes of several of these graphs. For the other parameters we refer to [24]. The mentioned paper also contains the weight enumerators of most of the codes. Here we restrict to the strongly regular graphs on 25 and 26 vertices, and the related regular two-graphs on 26 vertices. There are exactly four non-isomorphic regular two-graphs on 26 vertices with eigenvalues 2 and -3 . Together they have fifteen $SRG(25, 12, 5, 6)$'s (two from $LS(5)$'s one of which is the Paley graph) as a descendant and ten $SRG(26, 15, 8, 9)$'s

no.	(v, k, λ, μ)	a name	#graphs	#two-graphs	$\dim(C_A)$	$\dim(C_{\bar{A}})$
1	(5,2,0,1)	pentagon (Paley)	1	1	4	4
2	(9,4,1,2)	$L(3)$	1	1	4	4
3	(10,3,0,1)	Petersen ($\overline{T(5)}$)	1		6	4
4	(13,6,2,3)	Paley	1	1	12	12
5	(15,6,1,3)	$\overline{T(6)}$	1	1	14	4
6	(16,5,0,2)	Clebsch	1		16	6
7	(16,6,2,2)	$L(4)$	2		6	16
8	(17,8,3,4)	Paley	1	1	8	8
9	(21,10,3,6)	$\overline{T(7)}$	1		14	6
10	(25,8,3,2)	$L(5)$	1		8	16
11	(25,12,5,6)	$LS(5)$	15	4	12	12
12	(26,10,3,4)	$\overline{STS(13)}$	10		12	14
13	(27,10,1,5)	Schläfli	1	1	26	6
14	(28,12,6,4)	$T(8)$	4		6, 8	28
15	(29,14,6,7)	Paley	41	6	28	28
16	(35,16,6,8)	$\overline{STS(15)}$	3854	227	6,...,14	34
17	(36,10,4,2)	$L(6)$	1		10	36
18	(36,14,4,6)	HJsub	180		8,...,14	36
19	(36,14,7,4)	$T(9)$	1		8	27
20	(36,15,6,6)	$LS(6)$	32548		36	6,...,16
21	(37,18,8,9)	Paley	≥ 6760	≥ 191	36	36
22	(40,12,2,4)	$GQ(3, 3)$	28		10,...,16	40

Table 1. Primitive strongly regular graphs on fewer than 45 vertices

name	dim	0	4	6	8	10	12	14	16	18	20	22
s1	12	1		50	225	880	1225	1050	550	100	15	
s2	12	1	10	37	279	712	1343	1140	432	124	15	3
s3	12	1	12	43	279	696	1331	1152	448	124	9	1
s4	12	1	4	54	213	868	1237	1062	546	96	15	
s5	12	1	4	66	225	832	1201	1098	582	84	3	
s6	12	1	3	51	213	876	1243	1056	538	96	18	1
s7	12	1		54	225	864	1225	1074	550	84	15	4
s8	12	1	6	32	291	728	1331	1122	436	132	15	2
s9	12	1	8	38	291	712	1319	1134	452	132	9	
s10	12	1	7	39	295	708	1313	1140	456	128	8	1
s11	12	1	5	41	303	700	1301	1152	464	120	6	3
s12	12	1	7	35	291	720	1325	1128	444	132	12	1
s13	12	1	6	36	295	716	1319	1134	448	128	11	2
s14	12	1	7	35	291	720	1325	1128	444	132	12	1
s15	12	1	6	44	303	692	1295	1158	472	120	3	2

Table 2. Weight enumerators of the codes of the $SRG(25, 12, 5, 6)$'s.

name	dim	0	4	5	6	7	8	9	10	11	12	13
ls11	14	1		10	65	190	325	740	1430	1826	2275	2660
st11	14	1	13		52	130	403	884	1144	1950	2483	2264
st12	14	1	13	24	52	130	403	788	1144	1950	2483	2408
ls21	14	1	4	14	69	190	309	724	1414	1826	2299	2684
ls22	14	1	4	10	69	190	309	740	1414	1826	2299	2660
st21	14	1	8	26	47	130	423	780	1164	1950	2453	2420
st22	14	1	8	22	47	130	423	796	1164	1950	2453	2396
st23	14	1	8	26	47	130	423	780	1164	1950	2453	2420
st24	14	1	8	10	47	130	423	844	1164	1950	2453	2324
st25	14	1	8	22	47	130	423	796	1164	1950	2453	2396

Table 3. Weight enumerators of the codes of the $SRG(26, 15, 8, 9)$'s.

(two from $STS(13)$'s) in the switching class, see [32] and [2]. The corresponding codes of the form \mathcal{C}_A have been generated and the weight enumerators are given in Table 2 and Table 3 (keeping the names and order from [32]; the lines give the partition into the four switching-equivalence classes (two-graphs)). All codes are non-isomorphic. In most cases this follows from the weight enumerator, but in some cases more information is needed; see [24].

It follows that also the four two-graph codes are non-isomorphic and by Theorem 3.3 we have that the ten graphs on 26 vertices give rise to just four non-isomorphic codes of the form $\mathcal{C}_{\bar{A}+J} (= \mathcal{C}_A^\perp + \langle \mathbf{1} \rangle)$. In other words, by deleting the words of odd weight, the ten codes of length 26 collapse to the four two-graph codes.

4. Divisible Design Graphs

In this section we generalize the concept of a (v, k, λ) -graph, and introduce graphs with the property that the neighborhood design is a divisible design.

Definition 4.1 A k -regular graph is a divisible design graph (DDG for short) if the vertex set can be partitioned into m classes of size n , such that two distinct vertices from the same class have exactly λ_1 common neighbors, and two vertices from different classes have exactly λ_2 common neighbors.

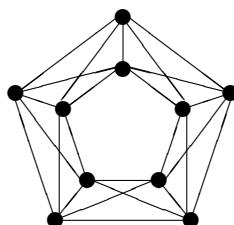


Figure 2. A proper divisible design graph

For example the graph of Figure 2 (which is the strong product of K_2 and C_5) is a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n) = (10, 5, 4, 2, 5, 2)$. Note that a DDG with $m = 1$, $n = 1$, or $\lambda_1 = \lambda_2$ is a (v, k, λ) graph. If this is the case, we call the DDG *improper*, otherwise it is called *proper*.

The definition of a divisible design (often also called group divisible design) varies. We take the definition given in Bose [8].

Definition 4.2 *An incidence structure with constant block size k is a (group) divisible design whenever the point set can be partitioned into m classes of size n , such that two points from one class occur together in λ_1 blocks, and two points from different classes occur together in exactly λ_2 blocks.*

A divisible design D is said to have the *dual property* if the dual of D (that is, the design with the transposed incidence matrix) is again a divisible design with the same parameters as D . From the definition of a DDG it is clear that the neighborhood design of a DDG is a divisible design D with the dual property. Conversely, a divisible design with a polarity with no absolute points is the neighborhood design of a DDG.

A DDG is closely related to a strongly regular graph. It follows easily that a proper DDG is strongly regular if and only if the graph or the complement is mK_n , the disjoint union of m complete graphs of size n .

Deza graphs (see [19]) are k -regular graphs which are not strongly regular, and where the number of common neighbors of two distinct vertices takes just two values. So proper DDGs, which are not isomorphic to mK_n or the complement, are Deza graphs.

4.1. Eigenvalues

With the identity matrix I_m of order m , and the $n \times n$ all-ones matrix J_n we define $K = K_{(m,n)} = I_m \otimes J_n = \text{diag}(J_n, \dots, J_n)$. Then we easily have that a graph Γ is a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if and only if Γ has an adjacency matrix A that satisfies:

$$A^2 = kI_v + \lambda_1(K_{(m,n)} - I_v) + \lambda_2(J_v - K_{(m,n)}). \quad (1)$$

Clearly $v = mn$, and taking row sums on both sides of Equation 1 yields

$$k^2 = k + \lambda_1(n - 1) + \lambda_2n(m - 1).$$

So we are left with at most four independent parameters. Some obvious conditions are $1 \leq k \leq v - 1$, $0 \leq \lambda_1 \leq k$, $0 \leq \lambda_2 \leq k - 1$. From Equation (1) strong information on the eigenvalues of A can be obtained. (Throughout we write eigenvalue multiplicities as exponents.)

Lemma 4.1 *The eigenvalues of the adjacency matrix of a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ are*

$$\left\{ k^1, \left(\sqrt{k - \lambda_1} \right)^{f_1}, \left(-\sqrt{k - \lambda_1} \right)^{f_2}, \left(\sqrt{k^2 - \lambda_2 v} \right)^{g_1}, \left(-\sqrt{k^2 - \lambda_2 v} \right)^{g_2} \right\},$$

where $f_1 + f_2 = m(n - 1)$, $g_1 + g_2 = m - 1$ and $f_1, f_2, g_1, g_2 \geq 0$.

Proof. The eigenvalues of $K_{(m,n)}$ are $\{0^{m(n-1)}, n^m\}$. Because I_v, J_v and $K_{(m,n)}$ commute it is straightforward to compute the eigenvalues of A^2 from equation (1). They are

$$\{(k^2)^1, (k - \lambda_1)^{m(n-1)}, (k^2 - \lambda_2 v)^{m-1}\},$$

and must be the squares of the eigenvalues of A . \square

Some of the multiplicities may be 0, and some values may coincide. In general, the multiplicities f_1, f_2, g_1 and g_2 are not determined by the parameters, but if we know one, we know them all because $f_1 + f_2 = m(n - 1)$, $g_1 + g_2 = m - 1$, and

$$\text{trace } A = 0 = k + (f_1 - f_2)\sqrt{k - \lambda_1} + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v}. \quad (2)$$

This equation leads to the following result.

Theorem 4.3 Consider a proper DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$, and eigenvalue multiplicities (f_1, f_2, g_1, g_2) .

- a. $k - \lambda_1$ or $k^2 - \lambda_2 v$ is a nonzero square.
- b. If $k - \lambda_1$ is not a square, then $f_1 = f_2 = m(n - 1)/2$.
- c. If $k^2 - \lambda_2 v$ is not a square, then $g_1 = g_2 = (m - 1)/2$.

Proof. If one of $k - \lambda_1$ and $k^2 - \lambda_2 v$ equals 0, then Equation (2) gives that the other one is a nonzero square. If $k - \lambda_1$ and $k^2 - \lambda_2 v$ are both non-squares, it follows straightforwardly that the square-free parts of these numbers are equal non-squares, hence Equation (2) has no solution. The second and third statement are obvious consequences of Equation (2). \square

If $k - \lambda_1$, or $k^2 - \lambda_2 v$ is not a square, the multiplicities (f_1, f_2, g_1, g_2) can be computed from the parameters. The outcome must be a set of nonnegative integers. This gives a condition on the parameters, which is often referred to as the *rationality condition*. Only if $k - \lambda_1$ and $k^2 - \lambda_2 v$ are both squares (that is, all eigenvalues of A are integers), the parameters do not determine the spectrum. Then $0 \leq g_1 \leq m - 1$, so there are at most m possibilities for the set of multiplicities.

4.2. The quotient matrix

The vertex partition from the definition of a DDG gives a partition (which will be called the *canonical partition*) of the adjacency matrix

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{bmatrix}.$$

We shall see that the canonical partition is equitable, which means that each block A_{ij} has constant row (and column) sum. For this, we introduce the $v \times m$ matrix S , whose columns are the characteristic vectors of the partition classes. Then S satisfies

$$S = I_m \otimes \mathbf{1}_n, \quad S^\top S = nI_m, \quad SS^\top = K_{(m,n)},$$

where $\mathbf{1}_n$ denotes the all-ones vector with n entries. Next we define $R = \frac{1}{n}S^\top AS$, which means that each entry r_{ij} of R is the average row sum of A_{ij} . We will call R the *quotient matrix* of A .

Theorem 4.4 *The canonical partition of the adjacency matrix of a proper DDG is equitable, and the quotient matrix R satisfies*

$$R^2 = RR^\top = (k^2 - \lambda_2 v)I_m + \lambda_2 n J_m.$$

The eigenvalues of R are

$$\left\{ k^1, \left(\sqrt{k^2 - \lambda_2 v} \right)^{g_1}, \left(-\sqrt{k^2 - \lambda_2 v} \right)^{g_2} \right\}.$$

Proof. Equation (1) gives $(\lambda_1 - \lambda_2)K_{(m,n)} = A^2 - \lambda_2 J_v - (k - \lambda_1)I_v$. Clearly A commutes with the right hand side of this equation and therefore with $K_{(m,n)}$. Thus $ASS^\top = SS^\top A$. Using this we find:

$$SR = \frac{1}{n}SS^\top AS = \frac{1}{n}ASS^\top S = AS,$$

which reflects that the partition is equitable. Similarly,

$$R^2 = \frac{1}{n^2}S^\top ASS^\top AS = \frac{1}{n}S^\top A^2 S = (k^2 - \lambda_2 v)I_m + \lambda_2 n J_m,$$

where in the last step we used $k^2 = k + \lambda_1(n-1) + \lambda_2 n(m-1)$. From the formula for R^2 it follows that R has eigenvalues $\pm\sqrt{k^2 - \lambda_2 v}$, whose multiplicities add up to $m-1$. If v is an eigenvector of R , then Sv is an eigenvector of A for the same eigenvalue. Therefore the multiplicity of an eigenvalue $\pm\sqrt{k^2 - \lambda_2 v}$ of R is at most equal to the multiplicity of the same eigenvalue of A . This implies that the multiplicities are the same. \square

The above lemma can easily be generalized to divisible designs with the dual property. This more general version of the lemma is due to Bose [8] (who gave a much longer proof).

If one wants to construct a DDG with a given set of parameters, one first tries to construct a feasible quotient matrix. For this the following straightforward properties of R can be helpful:

Proposition 4.1 *The quotient matrix R of a DDG satisfies*

$$\begin{aligned} \sum_i (R)_{i,j} &= k \text{ for } j = 1, \dots, m, \\ \sum_{i,j} (R)_{i,j}^2 &= \text{trace}(R^2) = mk^2 - (m-1)\lambda_2 v, \\ 0 \leq \text{trace}(R) &= k + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v} \leq m(n-1). \end{aligned}$$

In some cases these conditions lead to nonexistence or limited possibilities for R .

Proposition 4.2 *If $m = 3$ and $k^2 - \lambda_2 v$ is not a square, then the following system of equations has an integral solution.*

$$\begin{aligned} X + Y + Z &= k, \\ X^2 + Y^2 + Z^2 &= k^2 - 2\lambda_2 v/3, \\ X^3 + Y^3 + Z^3 &= 3XYZ + k(k^2 - \lambda_2 v). \end{aligned}$$

Proof. The quotient matrix R is a symmetric 3×3 matrix with all row and column sums equal to k and, since $k^2 - v\lambda_2$ is not a square, also $\text{trace}(R) = k$. This implies

$$R = \begin{bmatrix} X & Y & Z \\ Y & Z & X \\ Z & X & Y \end{bmatrix},$$

so $\text{trace}(R^2) = 3(X^2 + Y^2 + Z^2) = k^2 + 2(k^2 - \lambda_2 v)$. The third equation comes from $\det R = -k(k^2 - \lambda_2 v)$. \square

For example a DDG with parameters $(21, 12, 8, 6, 3, 7)$ does not exist because $X^2 + Y^2 + Z^2 = 60$ has no integral solution. Note that Construction 4.11 gives infinitely many DDGs that satisfy the condition of the above proposition.

Proposition 4.3 *There exists no DDG for the parameter sets $(14, 10, 6, 7, 7, 2)$, and $(20, 11, 2, 6, 10, 2)$.*

Proof. In both cases $n = 2$, so $\text{trace } R \leq m$. For the first parameter set this gives a contradiction, because $\text{trace } R = k = 10$ and $m = 7$. For the second parameter set, Theorem 4.5 implies that $R = J + P$ for some symmetric permutation matrix P . Therefore $\text{trace } R = 10$, P has zero diagonal, and the spectrum of R is $\{11, 1^4, -1^5\}$. This implies that the adjacency matrix has eigenvalues $11, 3^{f_1}, -3^{f_2}, 1^4$ and -1^5 where $f_1 + f_2 = 10$. This is impossible. \square

The following result is essentially due to Bose [8] (though his formulation is different).

Theorem 4.5 *Consider a DDG with parameters $(v, k, \lambda_1, \lambda_2, m, n)$. Write $k = mt + k_0$ for some integers t and k_0 with $0 \leq k_0 \leq m - 1$. Then the entries of R take exactly one, or two consecutive values if and only if*

$$k_0^2 - mk_0 - k^2 + km + \lambda_1 m(n - 1) = 0.$$

If this is the case then $R = tJ + N$, where N is the incidence matrix of a (possibly degenerate) (m, k_0, λ_0) design with a polarity.

Proof. If each entry of R equals t or $t + 1$, then in each row k_0 entries are equal to $t + 1$ and $m - k_0$ entries are equal to t (because the row sums of R are k). Therefore,

$$mk_0(t + 1)^2 + mt^2(m - k_0) = \text{trace}(R^2) = mk^2 + (m - 1)\lambda_2 v,$$

which leads to $k_0^2 - mk_0 - k^2 + km + \lambda_1 m(n - 1) = 0$. Conversely, if the equation holds, then a matrix R with k_0 entries $t + 1$ in each row, and all other entries equal to t satisfies the conditions of Equation 4.1. Moreover, any other solution to these equations has the same properties. (Indeed changing some entries to integer values different from t and $t + 1$, such that the sum of the entries remains the same, increases the sum of the squares of the entries). Suppose $R = tJ + N$ for some incidence structure N , then $N = N^\top$, and Theorem 4.4 implies that $N^2 \in \langle J, I \rangle$, therefore N is the incidence matrix of a (m, k_0, λ) design. \square

Note that the number of absolute points of the polarity equals $\text{trace } N = \text{trace } R - mt = k + (g_1 - g_2)\sqrt{k^2 - \lambda_2 v} - mt$, which is equal to $k - mt = k_0$ if $k^2 - \lambda_2 v$ is not a square.

4.3. Constructions

In this section we present some constructions of DDGs.

4.3.1. (v, k, λ) graphs and designs

We recall that the *incidence graph* of a design with incidence matrix N is the bipartite graph with adjacency matrix

$$\begin{bmatrix} O & N \\ N^\top & O \end{bmatrix}.$$

Construction 4.6 *The incidence graph of an (n, k, λ_1) design with $1 < k \leq n$ is a proper DDG with $\lambda_2 = 0$.*

Construction 4.7 *The disconnected graph for which each component is an (n, k, λ_1) graph ($1 < k < n$), or the incidence graph of an (n, k, λ_1) design ($1 < k \leq n$), is a proper DDG with $\lambda_2 = 0$.*

Proposition 4.4 *For a proper DDG Γ the following are equivalent.*

- a. Γ comes from Construction 4.6, or 4.7.
- b. Γ is bipartite or disconnected.
- c. $\lambda_2 = 0$.

Proof. It is clear that a bipartite or disconnected DDG has $\lambda_2 = 0$. Assume Γ is a DDG with $\lambda_2 = 0$. Then in every block row of the canonical partition of the adjacency matrix there is exactly one nonzero block (otherwise the neighborhood of a vertex contains vertices in different blocks which contradicts $\lambda_2 = 0$), and each nonzero block is the incidence matrix of a (n, k, λ_1) design. If such a block is on the diagonal it is the adjacency matrix of a (n, k, λ_1) graph with $1 < k < n$. If it is not on the diagonal the transposed block is on the transposed position, and together they make the bipartite incidence graph of a (n, k, λ_1) design with $1 < k \leq n$. \square

Construction 4.8 *If A' is the adjacency matrix of a (m, k', λ') graph ($1 \leq k' < m$), then $A' \otimes J_n$ is the adjacency matrix of a proper DDG with $k = \lambda_1 = nk'$, $\lambda_2 = n\lambda'$.*

Proposition 4.5 *For a proper DDG Γ the following are equivalent.*

- a. Γ comes from Construction 4.8.
- b. The adjacency matrix of Γ can be written as $A' \otimes J_n$ for some $m \times m$ matrix A' .
- c. $\lambda_1 = k$.

Proof. The only nontrivial claim is that c implies a. Assume Γ is a DDG with $k = \lambda_1$. Then any two rows of the adjacency matrix belonging to the same class are identical. Since the blocks have constant row and column sum this implies that all blocks have only ones, or only zeros. Therefore the adjacency matrix has the form $A' \otimes J_n$, where A' is a symmetric $(0, 1)$ -matrix with zero diagonal and row sum k/n . Moreover, any two distinct rows of A' have inner product λ_2/n . Therefore A' is the adjacency matrix of a (m, k', λ') graph. \square

Construction 4.9 Let A_1, \dots, A_m ($m \geq 2$) be the adjacency matrices of m (n, k', λ') graphs with $0 \leq k' \leq n - 2$. Then $A = J - K + \text{diag}(A_1, \dots, A_m)$ is the adjacency matrix of a proper DDG with $k = k' + n(m - 1)$, $\lambda_1 = \lambda' + n(m - 1)$, $\lambda_2 = 2k - v$.

Proposition 4.6 For a proper DDG Γ the following are equivalent.

- Γ comes from Construction 4.9.
- The complement of Γ is disconnected.
- $\lambda_2 = 2k - v$.

Proof. Let x and y be two vertices of Γ . Simple counting gives that the number of common neighbors is at most $2k - v$, and equality implies that x and y are adjacent. So, if $\lambda_2 = 2k - v$, then two vertices from different classes are adjacent, and hence the complement is disconnected. Conversely, suppose Γ is a DDG with disconnected complement \bar{G} (say). Let x and y be vertices in different components of \bar{G} . Then x and y have no common neighbors in \bar{G} , and hence x and y are adjacent vertices in Γ with $2k - v$ common neighbors. Therefore $\lambda_2 = 2k - v$, and all vertices from different classes are adjacent. Finally, equivalence of *a* and *b* is straightforward. \square

Note that in the above constructions the used (v, k, λ) graphs and designs may be degenerate. This means that the above constructions include the k -regular complete bipartite graph ($k \geq 2$), the $(k + 1)$ -regular complete bipartite graph minus a perfect matching ($k \geq 2$), the disjoint union of m complete graphs K_n ($m \geq 2, n \geq 3$), the complete m -partite graph with parts of size n ($m \geq 2, n \geq 2$), and the complete m -partite graphs with parts of size n extended with a perfect matching of the complement ($m \geq 2, n \geq 4, n$ even). So these DDGs exist in abundance, and we'll call them *trivial*.

4.3.2. Hadamard matrices

Construction 4.10 Consider a regular graphical Hadamard matrix H of order $m \geq 4$ and row sum $\ell = \pm\sqrt{m}$. Let $n \geq 2$. Replace each entry with value -1 by $J_n - I_n$, and each $+1$ by I_n , then we obtain the adjacency matrix of a DDG with parameters $(mn, n(m - \ell)/2 + \ell, (n - 2)(m - \ell)/2, n(m - 2\ell)/4 + \ell, m, n)$.

In terms of the adjacency matrix the construction becomes:

$$H \otimes I_n + \frac{1}{2}(J - H) \otimes J_n.$$

Using this, it is straightforward to check that Equation 1 is satisfied. We recall (see Section 2.3) the two regular graphical Hadamard matrices of order 4:

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}.$$

For the first one, the DDG is the $4 \times n$ grid, that is, the line graph of $K_{4,n}$. The second one gives DDGs with parameters $(4n, 3n - 2, 3n - 6, 2n - 2, 4, n)$; for $n = 2$ this is the complement of the cube. The DDGs of Construction 4.10 are improper whenever $\lambda_1 = \lambda_2$, which is the case if and only if $n = 4$.

Construction 4.11 Consider a regular graphical Hadamard matrix H of order $\ell^2 \geq 4$ with diagonal entries -1 and row sum ℓ . The graph with adjacency matrix

$$A = \begin{bmatrix} M & N & O \\ N & O & M \\ O & M & N \end{bmatrix}, \text{ where}$$

$$M = \frac{1}{2} \begin{bmatrix} J + H & J + H \\ J + H & J + H \end{bmatrix}, \text{ and } N = \frac{1}{2} \begin{bmatrix} J + H & J - H \\ J - H & J + H \end{bmatrix},$$

is a DDG with parameters $(6\ell^2, 2\ell^2 + \ell, \ell^2 + \ell, (\ell^2 + \ell)/2, 3, 2\ell^2)$.

For the two Hadamard matrices presented above, this leads to DDGs with parameters $(24, 10, 6, 3, 3, 8)$ and $(24, 6, 2, 1, 3, 8)$, respectively.

4.3.3. Divisible designs

Here we examine known constructions of divisible designs that admit a symmetric incidence matrix with zero diagonal, and therefore correspond to DDGs. Clearly, we can restrict ourselves to divisible designs with the dual property. Many constructions for these kind of designs come from divisible difference sets. Such a construction uses a group \mathcal{G} of order $v = mn$, together with a subset of \mathcal{G} of order k , called the base block. The blocks of the design are the images of the base block under the group operation. Thus we obtain v blocks of size k (blocks may be repeated). This construction gives a divisible design if the group \mathcal{G} has a normal subgroup \mathcal{N} of order n and the base block is a so called divisible difference set relative to \mathcal{N} . It follows from the construction that such a divisible design has the dual property. Moreover, one can order the points and blocks such that the incidence matrix becomes symmetric, and it is also easy to find an ordering that gives a zero diagonal. The problem is to find an ordering that simultaneously provides a symmetric matrix and a zero diagonal. Such an ordering is not always possible. For having a symmetric incidence matrix with zero diagonal, the divisible difference set should be reversible (or equivalently, it must have a strong multiplier -1). Several reversible relative difference sets are known. For example, for the group $\mathcal{G} = C_5 \times S_2 = \{1, a, a^2, a^3, a^4\} \times \{1, b\}$ the base block $\{(1, b), (a, 1), (a, b), (a^4, 1), (a^4, b)\}$ is a reversible difference set relative to $\mathcal{N} = S_2$, and hence gives a DDG. This DDG is the one given in Figure 2. In fact, several of the examples constructed so far can also be made with a reversible divisible difference set. These include all trivial examples and some of the ones from Construction 4.10. For more examples and information on reversible difference sets we refer to [1].

Another useful result on divisible designs is the construction and characterization of divisible designs with $k - \lambda_1 = 1$ given in [20]. We recall that the *strong product* of two graphs with adjacency matrices A and B , is the graph with adjacency matrix $(A + I) \otimes (B + I) - I$.

Construction 4.12 Let Γ' be a strongly regular graph with parameters $(m, k', \lambda, \lambda + 1)$. Then the strong product of K_2 with Γ' is a DDG with $n = 2$, $\lambda_1 = k - 1 = 2k'$ and $\lambda_2 = 2\lambda + 2$.

Checking the correctness of the construction is straightforward. There exist infinitely many strongly regular graphs with the required property. For example the Paley graphs. But there are infinitely many others. It easily follows that the complement of a strongly regular graph with $\mu - \lambda = 1$ has the same property. Thus we can get two DDGs from one strongly regular graph with $\mu - \lambda = 1$, unless the strongly regular graph is isomorphic to the complement (which is the case for the Paley graphs). For example the Petersen graph and its complement lead to DDGs with parameters $(v, k, \lambda_1, \lambda_2, m, n) = (20, 7, 6, 2, 10, 2)$ and $(20, 13, 12, 8, 10, 2)$, respectively. The pentagon, which is a strongly regular graph with parameters $(5, 2, 0, 1)$, leads once more to the example of Figure 2. In fact, several graphs coming from Construction 4.12 can also be constructed by use of a reversible divisible difference set. This includes all Paley graphs.

Theorem 4.13 *Let Γ be a nontrivial proper DDG, then Γ comes from Construction 4.12 if and only if $k - \lambda_1 = 1$.*

Proof. Assume Γ is a DDG with $k - \lambda_1 = 1$. According to [20] the neighborhood design D , or its complement has incidence matrix $N = (A \otimes J_n) + I_v$, where one of the following holds: (i) $J - 2A$ is the core of a skew-symmetric Hadamard matrix (this means that $A + A^\top = J - I$, and $4AA^\top = (v + 1)I + (v - 3)J$). (ii) $n = 2$, and A is the adjacency matrix of a strongly regular graph with $\mu - \lambda = 1$, or (iii) $A = O$, or $A = J - I$. Case iii and its complement correspond to trivial DDGs. Case ii corresponds to Construction 4.12 (note that N has no zero diagonal, but interchanging the two rows in each class gives N the required property). Also the complement of Case ii corresponds to Construction 4.12. Indeed, $J_v - N = J_v - (A \otimes J_2) - I_v = (J_m - A) \otimes J_2 - I_v$, where A , and therefore also $J_m - A - I_m$ is the adjacency matrix of a strongly regular graph with $\mu - \lambda = 1$. Finally we will show that Case i is not possible for a DDG. Suppose $PN = P(A \otimes J) + P$, or $P(J - N)$ is symmetric with zero diagonal for some permutation matrix P , then P is symmetric and preserves the block structure. The quotient matrix Q of P is a symmetric permutation matrix such that QA is symmetric with zero diagonal. We have $A + A^\top = J - I$, so $J - Q = AQ + A^\top Q = AQ + QA$, and therefore $\text{trace}(J - Q) = 2 \text{trace}(QA) = 0$, so $Q = I$, a contradiction. \square

4.3.4. Partial complements

The complement of a DDG is almost never a DDG again. If the partition classes are the same, then only the complete multipartite graph and its complement have this property. The cube (which is a bipartite DDG with two classes) and its complement (which is a DDG with four classes) is an example where the canonical partitions differ. However, if we only take the complement of the off-diagonal blocks it is more often the case that we get a DDG again. We call this the *partial complement* of the DDG. We have seen one such example in Construction 4.12, where the partial complement can be constructed in the same way, and hence produces no new examples. The following idea however can give new examples.

Proposition 4.7 *The partial complement of a proper DDG Γ is again a DDG if one of the following holds:*

- a. *The quotient matrix R equals $t(J - I)$ for some $t \in \{1, \dots, n - 1\}$.*
- b. *$m = 2$.*

Proof. We use Equation 1. In Case *a*, the partial complement has adjacency matrix $\tilde{A} = J - K - A$. In Section 4.2 we saw that $AK = KA = ASS^T = SRS^T$. Since $R = t(J - I)$ this implies $AK \in \text{Span}\{J, K\}$. Therefore $\tilde{A}^2 \in \text{Span}\{I, J, K\}$, and \tilde{A} represents a DDG.

In Case *b*, the vertices can be ordered such that the partial complement has adjacency matrix $\tilde{A} = J - K + DAD$, where $D = \text{diag}(1, \dots, 1, -1, \dots, -1)$. The quotient matrix R is a symmetric 2×2 matrix with constant row sum, hence $R \in \text{Span}\{I_2, J_2\}$, and therefore $AK = SRS^T \in \text{Span}\{K_{2,n}, J_v\}$, and also $DADK = DAK \in \text{Span}\{K_{2,n}, J_v\}$. Moreover, $(DAD)^2 = DA^2D \in \text{Span}\{I_v, J_v, K_{2,n}\}$, and hence $\tilde{A}^2 \in \text{Span}\{I, J, K\}$, which proves our claim. \square

Taking partial complements often gives improper DDGs. Conversely, the arguments also work if Γ is an improper DDG (that is, Γ is a (v, k, λ) graph), provided Γ admits a non-trivial equitable partition that satisfies *a* or *b*. An equitable partition of a (v, k, λ) graph that satisfies *a* is a so called Hoffman coloring (see [25]). Note that the diagonal blocks are zero, so the partition corresponds to a vertex coloring. Thus we have:

Construction 4.14 *Let Γ be a (v, k, λ) graph. If Γ has a Hoffman coloring, or an equitable partition into two parts of equal size, then the partial complement is a DDG.*

Also this construction can give improper DDGs, but in many cases the DDG is proper. For example there exists a strongly regular graph Γ with parameters $(v, k, \lambda, \mu) = (40, 12, 2, 4)$ with a so called spread, which is a partition of the vertex set into cliques of size 4 (see [25]). The complement of Γ is a $(40, 27, 18)$ graph, and the spread of Γ is a Hoffman coloring in the complement. The partial complement is Γ with the edges of the cliques of the spread removed. This gives a DDG with parameters $(40, 9, 0, 2, 10, 4)$. By taking the union of five classes in this Hoffman coloring, we obtain an equitable partition into two parts of size 20. The partial complement with respect to this partition gives a DDG with parameters $(40, 17, 8, 6, 2, 20)$.

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